A10289W1

FIRST PUBLIC EXAMINATION

Long Vacation

Preliminary Examination in Physics

Paper CP4: MATHEMATICAL METHODS 2

also

Preliminary Examination in Physics and Philosophy

Thursday 6 September 2018, 9.30 am – 12.00 noon

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a new page.

The use of approved calculators is permitted.

A list of physical constants, mathematical formulae and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. Find the limits

(a) $\lim_{x\to 1} \dfrac{x^6 - x}{3(x-1) - \arcsin(x-1)}$, [3]

(b) $\lim_{x\to\infty} \dfrac{\ln x \sin(x^2)}{x}$. [3]

2. Evaluate the derivative of $f(x) = (\cot x)^x$ at $x = \pi/4$. [3]

3. Find the Taylor expansion of $\sin x$ around $x = 0$ to fifth order in x.

Using the Taylor expansion of $\sin x$, evaluate the integral $\int_1^\infty x^{-1/2} \sin x \, \mathrm{d}x$ with two-digit accuracy. [6]

[Hint: you may use $\int_0^\infty x^{-1/2} \sin x \, \mathrm{d}x = \sqrt{\pi/2}$ without proof.]

4. Find the indefinite integrals

(a) $\int (\sin x)^5 (\cos x)^3 \, \mathrm{d}x$, [3]

(b) $\int \dfrac{\mathrm{d}x}{1 + \sin x}$. [4]

[Hint: for the integral in (b), you may use the change of integration variable $t = \tan(x/2)$.]

5. Evaluate the double integral

$$I = \int\int_D e^{-x^2-y^2} \, \mathrm{d}x \, \mathrm{d}y,$$

where D is the area defined by $x^2 + y^2 \leq 1$, $x \geq 0$ and $x \leq y \leq \sqrt{3}x$. [5]

6. Find the projection on the $x - y$ plane of the intersection between the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + (y-1)^2 + z^2 = 1$.

Calculate the area A of the surface of the cone $z = \sqrt{x^2 + y^2}$ enclosed by the plane $z = 0$ and the sphere $x^2 + (y-1)^2 + z^2 = 1$. [8]

7. Two waves $\cos(kx - \omega(k)t)$ and $\cos(k'x - \omega(k')t)$ satisfy the dispersion relation $\omega(k)$, where $k' = k + \delta k$ and $\delta k \ll k$. Use the superposition of these two waves to explain the concepts of phase velocity and group velocity.

Calculate the group velocity of a wave that satisfies $\omega^2 = k^2c^2 + \omega_p^2$, where c and ω_p are positive constants. [5]

Section B

8.

(a) Consider the function

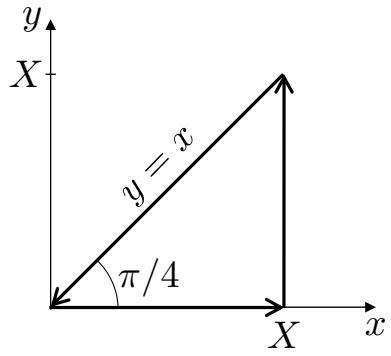
$$f(x,y,z) = x + \frac{1}{\sqrt{x^2+y^2+z^2}}.$$

Without using the divergence theorem, evaluate the surface integral $\int_B \nabla f \cdot \mathrm{d}\mathbf{S}$, where B is the sphere defined by $x^2+y^2+z^2 \leq \rho^2$. [5]

Show that $\nabla^2 f = 0$. The divergence theorem and the fact that $\nabla^2 f = 0$ would seem to indicate that the integral $\int_B \nabla f \cdot \mathrm{d}\mathbf{S}$ should vanish. Why is this integral not zero? [3]

(b) Let $\mathbf{F}(x,y,z) = (\sin(x^2-y^2)e^{-2xy}, \cos(x^2-y^2)e^{-2xy}, 0)$ be a vector field. Using Stoke's theorem, show that the line integral $\int_\Gamma \mathbf{F}\cdot\mathrm{d}\mathbf{r}$ vanishes for any closed loop Γ. [3]

Now let Γ be the triangle in the plane $z=0$ shown in the following figure:



Given that $\int_\Gamma \mathbf{F}\cdot\mathrm{d}\mathbf{r} = 0$, show that

$$\int_0^X \sin(x^2)\,\mathrm{d}x + I(X) = \int_0^X e^{-2x^2}\,\mathrm{d}x,$$

where X is the positive constant defined in the figure above, and

$$I(X) = \int_0^X \cos(X^2-x^2)e^{-2Xx}\,\mathrm{d}x.$$ [3]

Show that $|I(X)| \leq 1/2X$ and hence prove that

$$\int_0^\infty \sin(x^2)\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$ [6]

[Hint: you may use $\int_0^\infty e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}/2$ without proof.]

9. Explain how to determine the nature of the stationary points of a 2D function $f(x, y)$. [4]

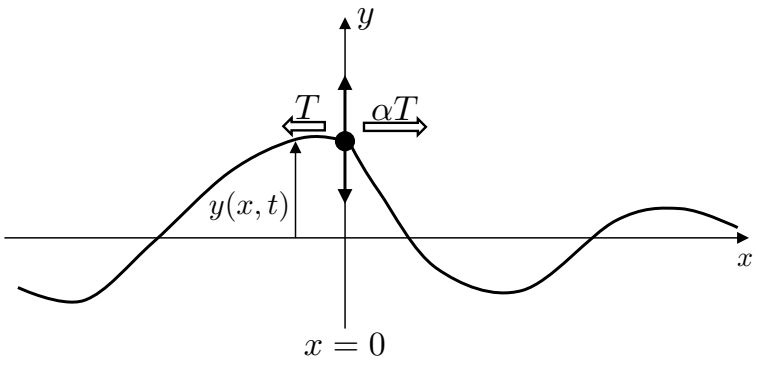
Find the five stationary points of the function $f(x, y) = (x^2 - y^2)e^{-x^2-y^2}$ and determine their nature. [10]

Sketch the contours of $f(x, y)$.

Find an exact expression for the contour $f(x, y) = 0$.

Determine the range of values that $f(x, y)$ can take. [6]

10. Consider the infinite taut string with linear density μ shown in the figure. At $x = 0$, the string is attached to a rail that allows the string to move up and down without friction. For $x < 0$, the tension on the string is T, and for $x > 0$, it is αT.



(a) Show that one of the boundary conditions at $x = 0$ is

$$\left.\frac{\partial y}{\partial x}\right|_{x=0^-} = \alpha \left.\frac{\partial y}{\partial x}\right|_{x=0^+}.$$

What is the other boundary condition? [2]

(b) A sinusoidal wave of frequency ω and amplitude A moves towards $x = 0$ from $x < 0$. Determine the amplitude of the reflected and transmitted waves. Check that energy is conserved. [6]

Now consider the situation in which there is a frictional force $-\beta T v_y$ between the string and the rail, where β is a constant and v_y is the vertical velocity of the string at $x = 0$.

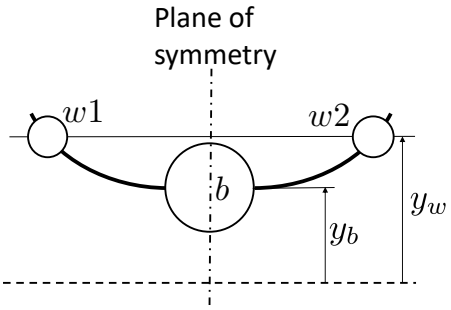
(c) Show that one of the boundary conditions at $x = 0$ is

$$\alpha \left.\frac{\partial y}{\partial x}\right|_{x=0^+} - \left.\frac{\partial y}{\partial x}\right|_{x=0^-} = \beta \left.\frac{\partial y}{\partial t}\right|_{x=0}. \quad [1]$$

(d) A sinusoidal wave of frequency ω and amplitude A moves towards $x = 0$ from $x < 0$. Determine the amplitude of the reflected and transmitted waves. Explain what happens in the limits $\beta = 0$ and $\beta \to \infty$. [7]

(e) Find an expression for the power dissipated by friction as a function of β, α, T, μ, ω and A. [4]

11. Consider the model of an aircraft body and its wings in the following figure:



Let y_b be the vertical displacement of the aircraft body b, and let y_{w1} and y_{w2} be the vertical displacements of the two wings $w1$ and $w2$. Consider only symmetric motion such that both wings move in unison, $y_{w1} = y_{w2} = y_w$. The mass of the aircraft body is M, and the mass of each wing is $m < M$. In this simple model, the equations of motion are given by

$$\begin{aligned} M\frac{\mathrm{d}^2 y_b}{\mathrm{d}t^2} &= -2k(y_b - y_w), \\ m\frac{\mathrm{d}^2 y_w}{\mathrm{d}t^2} &= k(y_b - y_w), \end{aligned}$$

where k is the force constant.

(a) Calculate the normal mode frequencies and amplitudes. Why is one of the characteristic frequencies zero? [10]

(b) Atmospheric turbulence acts as a vertical force $F_y = F_0 \cos(\Omega t)$ applied on the body and both wings, that is,

$$\begin{aligned} M\frac{\mathrm{d}^2 y_b}{\mathrm{d}t^2} &= -2k(y_b - y_w) + F_y, \\ m\frac{\mathrm{d}^2 y_w}{\mathrm{d}t^2} &= k(y_b - y_w) + F_y. \end{aligned}$$

Ignoring the transient response that depends on the initial conditions, determine the motion of the body and the wings in the presence of turbulence.

For $\Omega = \sqrt{10k/M}$, show that the oscillation amplitude of the aircraft body b is

$$A_b = \frac{F_0}{10k}\frac{|3 - 10m/M|}{|1 - 8m/M|}.$$ [6]

(c) The fuel deposits are located inside the wings. As the fuel is gradually spent during the flight, the mass m of the wings decreases. At the start of the flight, $m = M/3$ and at the end, $m = M/10$. Sketch, as a function of m/M, the oscillation amplitude of the aircraft body A_b due to atmospheric turbulence with frequency $\Omega = \sqrt{10k/M}$. Explain what happens from the start to the end of the flight. [4]