A10289W1

FIRST PUBLIC EXAMINATION

Trinity Term

Preliminary Examination in Physics

Paper CP4: MATHEMATICAL METHODS 2

also

Preliminary Examination in Physics and Philosophy

Thursday 7 June 2018, 2.30 pm – 5.00 pm

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a new page.

The use of approved calculators is permitted.

A list of physical constants, mathematical formulae and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. State Stokes' theorem.

Given $\mathbf{F}(x, y, z) = (y - z, z - x, x - y)$, calculate the line integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{l}$, where C is any curve on the plane $x - z = 1$. [5]

2. Calculate the length of the curve $\mathbf{r}(t) = (\cos t, \sin t, t)$ for $0 < t < 2\pi$. [4]

3. Find the limit

$$\lim_{x\to\infty} \frac{\sqrt{x^3 + 10x^2} - \sqrt{x^3 + 2x^2 - 2x}}{\sqrt{6x + 3}}.$$ [5]

4. Sketch the curves $y = \sqrt{x^2 + 1}$ and $y = 2/\sqrt{x^2 + 1}$ for $x > 0$.

Using the substitution $x = \sinh t$ or otherwise, calculate the area between these two curves in the region $0 < x < 1$. [6]

5. Calculate $\mathrm{d}y/\mathrm{d}x$ and $\mathrm{d}z/\mathrm{d}x$ given that $y(x)$ and $z(x)$ satisfy the equations

$$\begin{aligned} x^2 + y^3 - y^2 - z^2 + 1 &= 0, \quad \text{and} \\ 2x + a(y^3 - y^2) + z^2 &= 0, \end{aligned}$$

where a is a constant parameter.

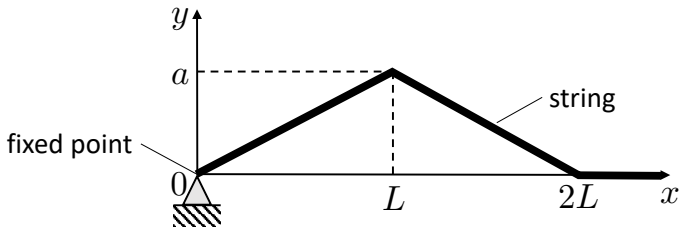
For which values of a do $\mathrm{d}y/\mathrm{d}x$ and $\mathrm{d}z/\mathrm{d}x$ become infinite? Explain geometrically why this is the case. [6]

6. Calculate the indefinite integral

$$\int \frac{\mathrm{d}\theta}{\cos\theta}.$$ [6]

7. Consider a semi-infinite taut string with linear density μ and tension T fixed at $x = 0$ and extending to $+\infty$. The vertical displacement from the equilibrium position of the string is $y(x, t)$.

The string is held at rest so that its vertical displacement $y(x, 0)$ is as shown in the diagram below.



At $t = 0$ the string is released. Let $y(x, t) = f(x - ct) + g(x + ct)$, where $c = \sqrt{T/\mu}$. Obtain $f(u)$ and $g(v)$ for $u > 0$ and $v > 0$ using the initial condition. Also obtain $f(u)$ for $u < 0$ using the boundary condition at $x = 0$.

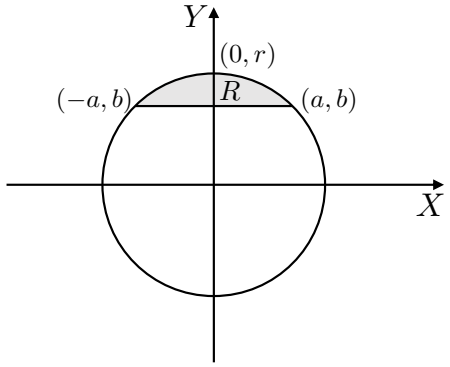
Plot the displacement $y(x, t)$ for $t = L/c$ and $t = 2L/c$. [8]

Section B

8. Find the stationary points of $f(x,y) = x^2 - y^2 - x^4 - y^4$. Determine whether they are maxima, minima or saddle points. [6]

Sketch the contour plot of $f(x,y)$. [4]

The region R consists of the points that satisfy $f(x,y) > 0$ and are in the quadrant $x > 0$ and $y > 0$. Show that, in the variables $X = x^2 - 1/2$ and $Y = y^2 + 1/2$, the region R is enclosed by a circle of radius r and a horizontal line (see figure below). In the figure, the coordinates a and b give the intersections of the circle and the line. Determine r, a and b. [5]



Using the integration variable substitution $X = x^2 - 1/2$ and $Y = y^2 + 1/2$, calculate the integral $\int_R xy \, \mathrm{d}x \, \mathrm{d}y$. [5]

9. (a) Consider a vector field $\mathbf{J}(\mathbf{r})$ that satisfies $\nabla \cdot \mathbf{J} = 0$ and is zero outside of a volume V. Show that, for any constant vectors $\mathbf{C}$ and $\mathbf{C}'$,

$$\int_V (\mathbf{C} \cdot \mathbf{r})(\mathbf{C}' \cdot \mathbf{J}) \, \mathrm{d}^3 r = -\int_V (\mathbf{C}' \cdot \mathbf{r})(\mathbf{C} \cdot \mathbf{J}) \, \mathrm{d}^3 r.$$

[Hint: you may use the divergence theorem and the vector field $(\mathbf{C} \cdot \mathbf{r})(\mathbf{C}' \cdot \mathbf{r})\mathbf{J}(\mathbf{r})$.] [8]

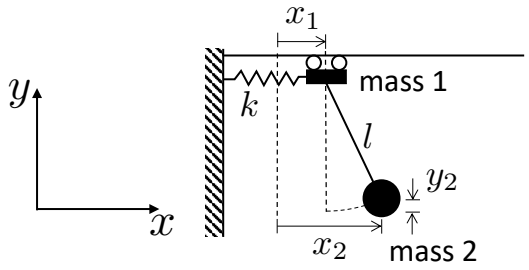
(b) Using the result in (a), prove that, for any constant vector $\mathbf{C}$,

$$\int_V (\mathbf{C} \cdot \mathbf{r})\mathbf{J} \, \mathrm{d}^3 r = \frac{1}{2}\mathbf{C} \times \int_V (\mathbf{J} \times \mathbf{r}) \, \mathrm{d}^3 r.$$ [4]

(c) Verify the result in (b) for $\mathbf{C} = (1,0,0)$ and

$$\mathbf{J}(x,y,z) = \begin{cases} (y,-x,0) & \text{for } x^2+y^2+z^2 < 1 \\ (0,0,0) & \text{for } x^2+y^2+z^2 \geq 1 \end{cases}.$$ [8]

10. Consider the motion of the two masses 1 and 2, each with mass m, in the vertical $x-y$ plane shown in the following figure:



Mass 1 is attached to the wall by a spring with spring constant k and is constrained to move in the x direction. Mass 2 is connected to mass 1 by a massless and inextensible string of length l.

Let x_1 and x_2 be the horizontal displacements of masses 1 and 2, respectively, from their equilibrium positions. Let y_2 be the vertical displacement of mass 2 from its equilibrium position.

(a) Assuming that the displacements of the masses from their equilibrium positions are small, show that the equations for x_1 and x_2 are given by

$$\begin{aligned} \frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} &= -\left(\frac{k}{m}+\frac{g}{l}\right)x_1 + \frac{g}{l}x_2, \\ \frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} &= \frac{g}{l}x_1 - \frac{g}{l}x_2. \end{aligned}$$

Indicate clearly where you have used the assumption of small displacements. [4]

(b) Show that the characteristic frequencies of this system are

$$\omega_{\pm}^2 = \frac{k}{2m} + \frac{g}{l} \pm \sqrt{\frac{k^2}{4m^2} + \frac{g^2}{l^2}}.$$ [5]

(c) Calculate the normal modes of the system and normalize them such that the amplitude of $x_1(t)$ is 1. [5]

(d) Under what initial conditions can the motion of mass 1 be described by $x_1(t) \propto \sin(\Omega t)\sin(\omega t)$? Under these initial conditions, determine the displacement of mass 2, $x_2(t)$. Give the frequencies Ω and ω assuming that $\Omega > \omega$. [6]

11. The field $\phi(x,t)$ satisfies the equation

$$\frac{\partial^3\phi}{\partial t^3} + \mathrm{i}\omega_n \frac{\partial^2\phi}{\partial t^2} - c^2\frac{\partial^3\phi}{\partial t \partial x^2} - \mathrm{i}\omega_d c^2 \frac{\partial^2\phi}{\partial x^2} = 0, \tag{1}$$

where c, ω_n and ω_d are positive constants. Show that this equation has wave solutions of the form $\phi(x,t) \propto e^{\mathrm{i}kx - \mathrm{i}\omega t}$, if ω and k satisfy the dispersion relation

$$k^2 = \frac{\omega^2(\omega - \omega_n)}{c^2(\omega - \omega_d)}.$$

[3]

Assuming that $\omega_n > \omega_d$, determine the values of ω for which the waves cannot propagate. [3]

Find the phase and group velocities in the limits $|\omega| \to 0$, $|\omega| \to \infty$, $\omega \to \omega_n$ and $\omega \to \omega_d$. [8]

Consider the case with $\omega_d = 0$ and $\omega_n = 3k_0c/2$, where k_0 is a constant. Using a solution of the form $\phi(x,t) = T(t)e^{\mathrm{i}k_0x}$, solve equation (1) with the initial conditions

$$\begin{aligned} \phi(x,0) &= e^{\mathrm{i}k_0x}, \\ \frac{\partial\phi(x,0)}{\partial t} &= 0, \\ \frac{\partial^2\phi(x,0)}{\partial t^2} &= -5k_0^2c^2e^{\mathrm{i}k_0x}. \end{aligned}$$

[6]