A10285W1

FIRST PUBLIC EXAMINATION

Long Vacation

Preliminary Examination in Physics

Paper CP3: MATHEMATICAL METHODS 1

also

Preliminary Examination in Physics and Philosophy

Wednesday 5 September 2018, 9.30 am – 12:00 noon

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a new page.

The use of calculators is **not** *permitted.*

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. (a) Express the term $(1+i)^4$ in the form $re^{i\theta}$, where r and θ are real variables.

(b) Express the complex number $\tan^{-1}(2i)$ in the form $x+iy$ where x, y are real.

(c) Given that $z = z_1 + z_2$, where $z_1 = e^{i\theta_1}$, $z_2 = e^{i\theta_2}$ and θ_1, θ_2 are real variables, find an expression for $|z|$ in terms of $\Delta\theta = \theta_1 - \theta_2$. [7]

2. (a) Write down the equation of the plane $3x + 4y + 5z = 10$ in the vector form

$$\mathbf{r} = \mathbf{r}_0 + t_1\mathbf{a} + t_2\mathbf{b} \, ,$$

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors in the plane, t_1 and t_2 are real parameters and $\mathbf{r}_0$ is a constant vector. What is the distance between the plane and the origin? [4]

(b) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be unit vectors. Show that

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c}) = \mathbf{b} \cdot \mathbf{c} - (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{c}) \, .$$

[4]

3. Consider the set of linear equations

$$\begin{aligned} 2x + y + z &= 2b \, , \\ ax + 3y + 2z &= 2a \, , \\ 2x + y + 3z &= 4 \, , \end{aligned}$$

where x, y, z are real variables and a and b are real parameters.

Find the values of a and b for which the set of equations have:
(i) a unique solution, (ii) infinitely many solutions, (iii) no solution. [5]

4. Find the general solution for the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 \frac{\mathrm{d}y}{\mathrm{d}x}\right) = n(n+1)y \, ,$$

where x is a real variable and n is a real constant. [5]

5. Let A and B be $n \times n$ hermitian matrices and U an $n \times n$ unitary matrix.

(a) Show that the modulus of each of the eigenvalues of U is equal to one ($|\lambda| = 1$).

(b) Show that the eigenvalues of A are real.

(c) Assuming that $U = A + iB$, show that

(i) $A^2 + B^2 = I$, where I is the identity matrix,

(ii) $AB - BA = 0$. [7]

6. The rotation matrix A in $\mathbb{R}^3$ is given by

$$A = \frac{1}{2} \begin{pmatrix} \sqrt{2} & -1 & -1 \\ 0 & \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & 1 & 1 \end{pmatrix} .$$

(a) Show that the matrix A is orthogonal.

(b) Calculate $\cos\theta$, where θ is the angle of rotation, and find a unit vector in the direction of the axis of rotation. [8]

Section B

7. (a) State de Moivre's theorem and show that

$$\text{(i)} \quad \sum_{n=0}^{N-1} \cos n\theta = \frac{\sin(N\theta/2)}{\sin(\theta/2)} \cos\frac{(N-1)\theta}{2} \ ,$$

$$\text{(ii)} \quad \cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta \ ,$$

where n, N are integers and θ is a complex variable. [8]

(b) Find all the roots for the equation

$$\frac{(z-1)^n}{(z+1)^n} = -1 \ . \qquad (\dagger)$$

Verify your solution for the special case $n = 3$, by finding the roots of the resulting third order equation.

Use the general solution to (†) to calculate the product $\displaystyle\prod_{r=1}^{n} \cot\frac{(2r+1)\pi}{2n}$, where r is an integer, for both odd and even values of n. [8]

(c) Show that if the complex numbers z and u satisfy the relation $\left|\dfrac{z+u}{z+u^\star}\right| = 1$, then either u or z must be real. [The $(\star)$ stands for the complex conjugate]. [4]

8. The differential equation for the displacement $y(t)$ of a particle executing forced and damped harmonic oscillations with damping factor γ and natural frequency ω_0 may be written as

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\gamma\frac{\mathrm{d}y}{\mathrm{d}t} + \omega_0^2 y = F\cos\omega t \ ,$$

where F and ω are the amplitude and frequency of the driving force respectively.

(a) Assuming that $F = 0$, find the displacements $y(t)$ of the particle for the cases $\omega_0 < \gamma$ and $\omega_0 = \gamma$. Sketch and compare the two displacements. [6]

(b) Assume now that $F = F_0 \neq 0$ and $\gamma < \omega_0$. Explain what is meant by a steady state solution of the differential equation and find an expression for the steady state amplitude and phase of the displacement. [6]

For a given value of the natural frequency ω_0, which value of the driving force frequency ω maximises the displacement? For what value of ω is the velocity a maximum? [4]

(c) Explain what is meant by the width of the oscillator resonance. Calculate the width of the resonance for the case $\gamma \ll \omega_0$. (You may assume that for very high quality factors the amplitude of oscillation is symmetric about ω_0). [4]

9. The matrix

$$A = \begin{pmatrix} 1 & -1 & \alpha \\ \alpha - 3 & 0 & 1 \\ 2 & -1 & \alpha + 1 \end{pmatrix}$$

defines the linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ by $f(x) = A\mathbf{x}$ where $\mathbf{x} \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$.

Let A_4 be equal to the matrix A for $\alpha = 4$.

(a) Find a basis for $\mathrm{Ker}(f)$ and show that the geometry of the kernel is a straight line in $\mathbb{R}^3$. Write the equation of the line in vector form. [4]

(b) By choosing $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ as a basis of $\mathbb{R}^3$ with $\mathbf{e}_1 \in \mathrm{Ker}(f)$, show that the geometry of the image of f is a plane and find the direction of the normal to this plane. [5]

Now let A_0 be equal to A for $\alpha = 0$.

(c) Assuming that the matrix A_0 was calculated with respect to the basis $\left(\mathbf{u}_1 = (1,0,0)^T, \mathbf{u}_2 = (0,1,0)^T, \mathbf{u}_3 = (0,0,1)^T\right)$, express the map $f(\mathbf{x})$ in terms of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ and the coordinates of the vector $\mathbf{x} = (x_1, x_2, x_3)^T$. [3]

(d) Assume now that the matrix A_0' of the map f is calculated with respect to the basis $\left(\mathbf{w}_1 = (1,1,0)^T, \mathbf{w}_2 = (1,0,1)^T, \mathbf{w}_3 = (0,1,1)^T\right)$. Calculate the matrix A_0' from the relation $A_0' = CA_0C^{-1}$ by a suitable choice of the matrix C. [8]

10.

(a) The components of a vector $\mathbf{a}$ in the (x, y) plane of a cartesian coordinate system (x, y, z) are given by (a_x, a_y). Assume now that the (x, y) axes are rotated about the z axis by an angle θ, anticlockwise, so that the components of $\mathbf{a}$ with respect to the rotated coordinate system (x', y', z) are given by $(a_{x'}, a_{y'})$. Calculate the elements of the matrix R that relates the vector $(a_{x'}, a_{y'})^T$ to the vector $(a_x, a_y)^T$ and show that R is a rotation matrix. Find the eigenvectors of the matrix R. [7]

(b) The equation of a conical section in the (x, y) coordinate system in $\mathbb{R}^2$ is given by:

$$f(x, y) = x^2 + 6xy + y^2 = 4. \qquad (\star)$$

Write down the above equation in the matrix form $\mathbf{x}^T M\mathbf{x} = 4$ where M is a symmetric matrix and $\mathbf{x} = (x, y)^T$ is a coordinate vector in $\mathbb{R}^2$. Use matrix diagonalisation to show that the curve in $(\star)$ represents a hyperbola. Find the elements of the unitary matrix that diagonalises the matrix M. [8]

Sketch this curve showing the asymptotes and the points of intersection with the (x, y) axes. [5]