A10285W1

FIRST PUBLIC EXAMINATION

Trinity Term

Preliminary Examination in Physics

Paper CP3: MATHEMATICAL METHODS 1

also

Preliminary Examination in Physics and Philosophy

Wednesday 6 June 2018, 2.30 pm – 5.00 pm

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a new page.

The use of calculators is **not** *permitted.*

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. Let $z = x + iy$ be a complex variable with real x, y. Show that

(a) $\tanh^{-1} z = \dfrac{1}{2} \ln \dfrac{1+z}{1-z}$

(b) $|\sin(z)| \geq |\sin(x)|$. [5]

2. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ be vectors in $\mathbb{R}^3$ and M be a 3×3 matrix.

(a) Prove the vector identity: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{c}$ [4]

(b) Find a linear relation between $\mathbf{c}$ and $\mathbf{d}$ such that the expression

$$a_i d_j b_k M_{ij} - a_i b_i d_j M_{kj}$$

becomes a component of the triple vector product in (a).

[The Einstein summation convention is used in the above expression.] [3]

3. The differential equation for the displacement $y(t)$ of a particle which is executing damped harmonic oscillations with a damping factor γ and natural frequency ω_0 may be written as

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \gamma \frac{\mathrm{d}y}{\mathrm{d}t} + \omega_0^2 y = 0 \, .$$

Show that for very light damping ($\gamma \ll \omega_0$), the displacement as a function of time may be written as

$$y(t) = y_0 \mathrm{e}^{-\frac{\gamma}{2}t} \cos(\omega_0 t + \alpha) \, ,$$

where α and y_0 are constants.

Calculate the total energy of the oscillator in this approximation, and show that it decreases exponentially with time. Hence show that the quality factor is given by $Q = \omega_0/\gamma$. [7]

4. Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)y^n \, ,$$

where x is a real variable and n is an integer.

(a) Show that the equation can be transformed into a linear first order differential equation using a suitable substitution.

(b) Find the general solution $y(x)$ for the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = x^3 y^n \, ,$$

for $n = 3$ and $n = 1$. [7]

5. Let A, B and C be 2×2 matrices and define the commutator $[A, B] = AB - BA$. Show that

(a) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$,

(b) if $[A, B] = 0$ and the eigenvalues of A are non-degenerate, then an eigenvector of A must also be an eigenvector of B,

(c) if A and B are hermitian matrices and $[A, B] = \alpha C$, where α is a scalar constant, then the matrix C can also be hermitian for a suitable choice of α. [7]

6. Let A be the 3×3 matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix} .$$

Find the eigenvalues of A and show that one of them is degenerate.

Find three linearly independent corresponding eigenvectors and indicate clearly which eigenvector belongs to which eigenvalue. Verify that the eigenvectors you have chosen are indeed linearly independent.

Use matrix row reduction to find the rank of the matrix $(A - \lambda I)$ where λ is the degenerate eigenvalue of the matrix A and I is the identity matrix. Hence explain why this result is expected given the degeneracy of the matrix A. [7]

Section B

7.

(a) The polynomial $f(z)$ where z is a complex variable is defined as

$$f(z) = z^4 - 2z^3 + 8z^2 - 8z + 16.$$

Show that the equation $f(z) = 0$ has two purely imaginary roots. Hence or otherwise find all the roots of the equation and show that the sum and product of the roots take the expected values. [7]

(b) Use de Moivre's theorem to show that

$$\tan 4\theta = \frac{4(u - u^3)}{1 + u^4 - 6u^2} \ ,$$

where $u = \tan\theta$. Use this expression to show that $\tan 22.5^\circ = \sqrt{3 - \sqrt{8}}$. [6]

(c) Explain why the equation of a plane in $\mathbb{R}^3$ may be written in the form $\mathbf{r} \cdot \mathbf{n} = d$, where $\mathbf{r}$ is the position vector of points on the plane. Use a sketch to define $\mathbf{n}$ and d.

Find the equation of the line of intersection of the two planes $2x + 6y - 3z = 10$ and $5x + 2y - z = 12$. Express the line equation in both the cartesian and vector forms. [7]

8.

(a) Find the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 60\cosh x \ .$$ [8]

(b) Consider the second order differential equation

$$(1 + x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6x\frac{\mathrm{d}y}{\mathrm{d}x} + 12y = 0.$$

Given that the solution can be expressed as a power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n \ ,$$

where n is an integer and a_n is a real constant, show that the expansion coefficients satisfy the relation

$$a_{n+2} = -\frac{(n-3)(n-4)}{(n+1)(n+2)}a_n \ .$$

[8]

Hence find the general solution $y(x)$ of the differential equation that satisfies the boundary conditions: $y(0) = 0$ and at $x = 0$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$. [4]

9. Let V be a vector space over the field F and $\phi : V \to V$ be a linear map.

(a) Assuming that the set of vectors $(\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n)$ forms a basis of V, explain how the elements of the matrix A representing the map ϕ are calculated from the image of the basis vectors. [3]

The matrix A' also corresponds to the linear map ϕ with respect to a different set of basis vectors $(\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_n)$. Show that the two matrices are related by the equation

$$A' = C^{-1}AC \ , \qquad (\dagger)$$

and explain how to calculate the matrix C.

Explain also how to calculate the elements of the matrix C for the matrix A' to be diagonal. [7]

(b) The velocity components of a particle moving in the (x, y) plane are given by the coupled differential equations:

$$\dot{v}_1 = \alpha v_2 \ ,$$

$$\dot{v}_2 = -\alpha v_1 \ ,$$

where the 'dot' indicates the derivative with respect to time ($\dot{v}_i = \dfrac{\mathrm{d}v_i}{dt}$). Find the matrix M that relates $(\dot{v}_1, \dot{v}_2)$ to the vector (v_1, v_2). [2]

Using the expression for the basis change given by the matrix equation $(\dagger)$, convert the differential equations above into non-coupled *first order* differential equations with new velocity components (v_1', v_2') and find expressions for $v_1'(t)$ and $v_2'(t)$.

By making a suitable transformation using the matrix C, find the solutions $(v_1(t), v_2(t))$ of the coupled differential equations that satisfy the boundary conditions $v_1(0) = 0$ and $v_2(0) = v_0$. [8]

10. Let V be a vector space over a field F and consider the scalar product $\langle \mathbf{v}, \mathbf{w} \rangle$ of two vectors $\mathbf{v}$, $\mathbf{w} \in V$.

(a) State the conditions that a symmetric scalar product ($F = \mathbb{R}$) and a hermitian scalar product ($F = \mathbb{C}$) must satisfy. [3]

(b) Show that the components v_i of a vector $\mathbf{v} \in \mathbb{C}^n$ with respect to an ortho-normal set of basis vectors $(\mathbf{e}_1, \mathbf{e}_2, ...\mathbf{e}_n)$ are given by $v_i = \langle \mathbf{e}_i, \mathbf{v} \rangle$, where $i = 1, 2, ..., n$. Hence use the properties of the scalar product in (a) to find an expression for the scalar product of two vectors $\langle \mathbf{u}, \mathbf{v} \rangle$ in $\mathbb{C}^n$, in terms of their components with respect to the basis vectors $(\mathbf{e}_1, \mathbf{e}_2, ...\mathbf{e}_n)$. [3]

(c) Consider the case $V = \mathbb{R}^n$ and let $(\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n)$ and $(\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_n)$ be two non-orthogonal basis sets that satisfy the relation

$$\langle \mathbf{u}_i, \mathbf{w}_j \rangle = \delta_{ij} \ , \qquad (*)$$

where δ_{ij} is the Kronecker delta.

Given that $\mathbf{a}$ is a vector with components a_i with respect to the basis vectors $(\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n)$ and $\mathbf{b}$ is another vector with components b_i with respect to the basis $(\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_n)$, use the definition in (a) to calculate the scalar product $\langle \mathbf{a}, \mathbf{b} \rangle$. [3]

(d) The components of $\mathbf{a}$ with respect to the basis $(\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_n)$ are given by a_i and with respect to the basis $(\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_n)$ are given by a'_i. By taking suitable scalar products and using the relation between the basis vectors in $(*)$, calculate the elements of the matrix $\mathbf{G}$ that relate the components a'_i to the components a_i. [4]

Now consider the special case of $V = \mathbb{R}^2$ and the basis vectors $(\mathbf{u}_1, \mathbf{u}_2)$ and $(\mathbf{w}_1, \mathbf{w}_2)$ that are related by equation $(*)$ in (c) for $n = 2$. Assuming that the basis vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are chosen as unit vectors with an angle of 60° between them, with $\mathbf{u}_1$ along the x axis, find the basis vectors $\mathbf{w}_1$ and $\mathbf{w}_2$.

Make a sketch in the (x, y) plane showing the two sets of components (a_1, a_2) and (a'_1, a'_2) of the vector $\mathbf{a}$ and use it to find the matrix that relates (a'_1, a'_2) to (a_1, a_2). Show that this result is consistent with the components obtained from the matrix $\mathbf{G}$ above. [7]