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FIRST PUBLIC EXAMINATION

Preliminary Examination in Physics

SECOND PUBLIC EXAMINATION

Honour School of Physics,
Parts A and B: 3 and 4 year Courses

SHORT OPTIONS

TRINITY TERM 2017

Tuesday, 13 June
9.30 am to 11.00 am for candidates offering ONE Short Option
9.30 am to 12.30 pm for candidates offering TWO Short Options

Answer **two** *questions from each option for which you have entered.*

Start the answer to each question in a fresh book.

The use of approved calculators is permitted.

If you have entered for two Short Options,
keep your answers to the two options in different books
and at the end hand in two bundles, one for each option.

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do NOT turn over until told that you may do so.

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Some sections start with a relevant rubric.

Section S1 FUNCTIONS OF A COMPLEX VARIABLE

1. A holomorphic function $f(z)$ may be written in terms of its real and imaginary parts as

$$f(z) = u(x, y) + \mathrm{i}v(x, y).$$

State the Cauchy-Riemann conditions satisfied by u and v and show that they imply that both u and v are harmonic functions. [5]

Show explicitly that

$$u(x, y) = \frac{1}{2}x \log(x^2 + y^2) - y \arctan \frac{y}{x}$$

is harmonic. Find a holomorphic function $f(z)$ such that $u = \mathrm{Re}f$ and determine $v = \mathrm{Im}f$. [9]

Show that the potential

$$V(x, y) = \frac{V_0}{\pi}\left(\frac{1}{2}y \log(x^2 + y^2) + x \arctan \frac{y}{x}\right)$$

solves the equations for electrostatics in the absence of charges and with boundary conditions

$$\begin{aligned} V(x, 0) &= V_0 x, \quad x \leq 0 \\ V(x, 0) &= 0, \quad x > 0. \end{aligned}$$

[4]

Show that the gradient of an equipotential line $y(x)$ satisfies

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\theta}{1 + \log r}$$

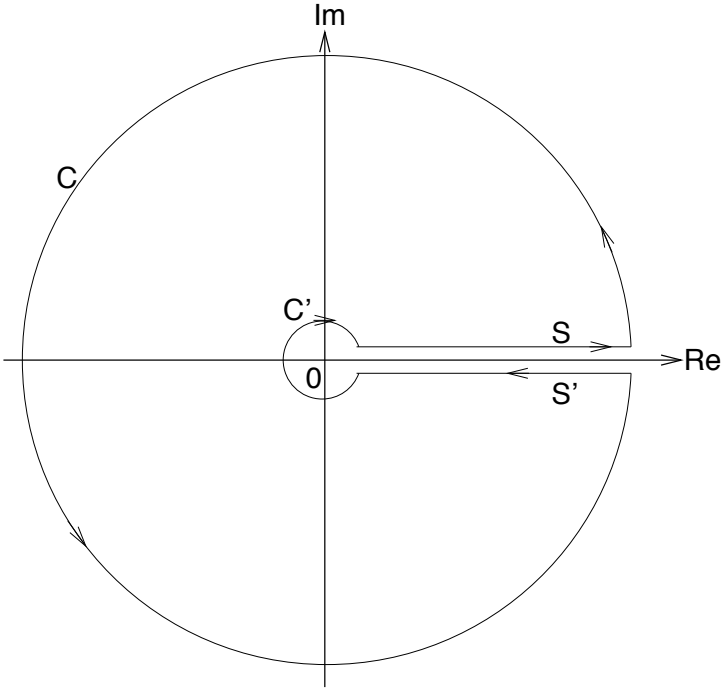
where $x + \mathrm{i}y = r\mathrm{e}^{\mathrm{i}\theta}$. For what ranges of (r, θ) is the gradient respectively positive, infinite, and negative? Sketch the equipotentials. [7]

2. What is meant by a branch point and a branch cut? Illustrate your answer using the function $\log z$. [4]

Show that

$$\oint_\Gamma \frac{(\log z)^n}{z^2+1}\mathrm{d}z = \pi^{n+1}\left(\frac{\mathrm{i}}{2}\right)^n (1-3^n) \qquad (1)$$

where n is a positive integer and the contour $\Gamma = S \cup C \cup S' \cup C'$ in the plane cut along the positive real axis as shown in the figure [5]



Defining

$$I_n = \int_0^\infty \frac{(\log x)^n}{x^2+1}\mathrm{d}x$$

use (1) with $n = 2$ to show that

$$I_0 = \frac{\pi}{2}, \qquad I_1 = 0.$$

[4]

Find I_2 and I_4. [12]

3. State Cauchy's Residue Theorem. [3]

Explain how the Residue Theorem can be used to calculate some definite integrals of the form

$$\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta$$

where $f(\theta) = f(\theta + 2\pi)$; for what forms of $f(\theta)$ is the method useful? [3]

Hence evaluate

$$\int_0^{2\pi} \frac{1}{1 + 2\tan\theta + 3\sec\theta}\,\mathrm{d}\theta.$$ [12]

Use the method of contour integration to evaluate

$$\int_{-\infty}^{\infty} \frac{\sin x}{x - \mathrm{i}a}\,\mathrm{d}x$$

where a is real and positive and hence show that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \pi.$$

[7]

Section S2 ASTROPHYSICS: FROM PLANETS TO THE COSMOS

1. State Kepler's third law of planetary motion and write it in equation form in either S.I. or other convenient units which you should define. [5]

A star of mass M_s exhibits a sinusoidal variation in its radial velocity of amplitude V_s and period P arising from the motion induced by a planetary companion. By assuming that the orbit of the planet is circular, edge-on to the observer and that the mass of the planet, M_p, is small compared to the star, show that:

$$M_p^3 \simeq V_s^3 \frac{PM_s^2}{2\pi G}.$$

[8]

A star on the Main Sequence with luminosity 3 $L_\odot$ exhibits a radial velocity variation of amplitude $30\,\mathrm{m\,s^{-1}}$ with period 10 days. Calculate the mass of the planet in Jupiter masses ($1.9 \times 10^{27}\,\mathrm{kg}$) and the radius of the planet's orbit in Astronomical Units.
The star is in the Orion star forming region a distance of 412 pc from the Sun. Calculate the angle subtended by the radius of the planet's orbit in arcseconds. Could this planet be resolved from its host star using Hubble Space Telescope? [12]

2. Write down the equation that defines the effective temperature of a star, identifying the other quantities in the equation. [4]

Star A has an apparent bolometric magnitude, $m_{bol} =$ -0.641 and has a parallax of 50 milliarcseconds. Its effective temperature is estimated to be 3900K. Make an estimate of the radius of the star in units of the solar radius. On which branch of the Hertzsprung-Russell diagram would you find star A? What is the source of energy for star A? [12]

Describe three other methods for estimating the surface temperature of a star. State the assumptions made in each case. Which of the three methods of estimating surface temperature of star A would be most accurate and why? [9]

[*Absolute bolometric magnitude of the sun:* $M_\odot^{bol}$*=4.72.*]

3. State Hubble's law as a vector equation relating the recession velocity of a galaxy, **v**, to its distance, **r**. By writing **r** in terms of the time independent co-moving distance **x**: **r**= R(t)**x**, where R is the scale factor of the expansion, show that Hubble's constant H is given by $H = \dot{R}/R$. [5]

The Friedmann equation, describing the expansion of a homogeneous, isotropic Universe in terms of the expansion factor, is:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{R^2} = H^2,$$

where ρ is the density of the Universe and k is a constant. Show that the critical density, ρ_c, in a flat Universe is given by:

$$\rho_c = \frac{3H^2}{8\pi G}.$$

Use the fact that the density, $\rho(t)$, is related to the expansion factor, $R(t)$, by $\rho = \frac{\rho_0}{R^3}$, where the subscript 0 refers to the present day value, to show that :

$$\dot{R}^2 = \frac{8\pi G\rho_0}{3R}.$$

Show that $R(t) = (t/t_0)^{\frac{2}{3}}$ is a solution to this equation and therefore, taking $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, calculate the age of the Universe in this model. [12]

The present mass density of the Universe is $\sim 2.5 \times 10^{-27}$ $\mathrm{kg\,m^{-3}}$ and the present day energy density in CMB photons is 5×10^5 $\mathrm{eV\,m^{-3}}$. Compare the energy density in matter with the energy density in the CMB radiation today, and calculate the redshift at which the two energy densities were equal. [8]

Section S3 QUANTUM IDEAS

1. State the four postulates which lead to Bohr's model of an atom. If one electron moves around one proton, show that the allowed orbital radii r_n are given by

$$r_n = \frac{4\pi\epsilon_0\hbar^2 n^2}{me^2} \ .$$

Explain all the terms in the above equation. [8]

(a) You may assume that the kinetic energy is half the magnitude of the potential energy. Derive the allowed values for the velocity of the electrons and their energies. [8]

(b) Find an expression for the wavelength of light when an electron makes a transition between the $n = 3$ and $n = 2$ level. [5]

(c) How are these expressions modified when the finite mass of the nucleus is taken into account? [4]

2. Write down Schrödinger's time independent equation for a particle in a potential well of height V explaining what the symbols denote. [4]

A particle of mass m is confined in an infinite potential well, where the potential is zero for $0 \leq x \leq L$ and infinite outside this range. Show that the wave functions of the form:

$$\psi = \sqrt{\frac{2}{L}} \sin(k_n x) \ ,$$

are solutions of the Schrödinger equation with energies given by:

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2 \pi^2 n^2}{2mL^2} \ .$$

What values can k_n take and why? Explain clearly why the wave functions are normalised. [10]

A particle starts in the lowest energy level. Imagine that the width of the well has suddenly doubled in size. What would the new energy of the particle be immediately after the doubling of the width? Calculate the probability that the particle stays in the new lowest energy level. [11]

3. Explain why the polarization of light can be treated as a quantum mechanical two-level system. [4]

Suppose that a single photon passes through a birefringent crystal. The horizontal polarization $|H\rangle$ is reflected more than the vertical polarization $|V\rangle$ and hence the two polarizations take two different paths inside the crystal and exist at two distinguishable spatial points, labeled by the kets $|\uparrow_1\rangle$ and $|\downarrow_1\rangle$. If the photon enters the crystal polarized at 45 degrees, write down the output state in terms of both the polarization states as well as the spatial states. [4]

What is the probability that the photon will be in the upper path?

Suppose that the photon has been detected in the lower path using a non-destructive detector. What is its polarization state after that detection? [5]

The lower path passes through a quarter-wave plate. Write down the total (polarization and spatial) state of the photon after this process.

The lower path now goes through another birefringent crystal of the same kind as the first one with the new outgoing paths being labeled as $|\uparrow_2\rangle$ and $|\downarrow_2\rangle$. Write down the final state of the photon including both the polarization and the space degrees of freedom.

What is the probability to detect the photon in the path $|\downarrow_2\rangle$? If the photon has been detected in this path, what is its polarization? [12]

Section S7 CLASSICAL MECHANICS

1. The motion of a charge in a magnetic field is governed by the Lagrangian

$$\mathcal{L} = \frac{1}{2}m\dot{\mathbf{r}}^2 + q\dot{\mathbf{r}} \cdot \mathbf{A}.$$

We consider motion in a simple axisymmetric field given by the vector potential $\mathbf{A} = A(r)\hat{\mathbf{z}}$ so that $\mathbf{B} = -\frac{\partial A}{\partial r}\hat{\boldsymbol{\theta}}$, where $r,\ \theta,\ z$ are cylindrical polar coordinates and $\hat{\boldsymbol{\theta}}$ is the unit vector in the θ direction.

(a) Show that the Lagrangian in cylindrical polars is:

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) + q\dot{z}A(r).$$

[8]

(b) Obtain expressions for the three canonical momenta - p_r, p_θ and p_z. Which of these momenta are constants of the motion? [6]

(c) Obtain an expression for the Hamiltonian. Is the Hamiltonian a constant of the motion? [6]

(d) We take $A(0) = 0$, $\frac{\partial A}{\partial r} > 0$ and $A(r \to \infty) \to \infty$. Show that a particle with $\dot{\theta} = 0$ at $t = 0$ and $p_z^2 > 2mH$ is confined between two radii r_1 and r_2, where $\sqrt{2mH} = p_z - qA(r_1)$ and $\sqrt{2mH} = qA(r_2) - p_z$. [2]

(e) For low energy particles with $2mH \ll p_z^2$ and $p_z = qA(r_0)$ show that

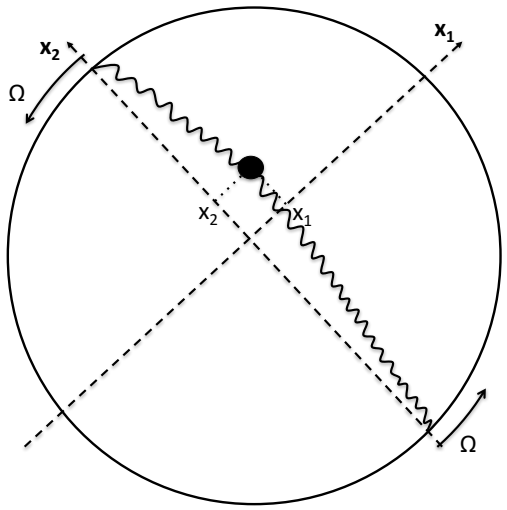
$$\dot{s}^2 + \Omega^2 s^2 = \Omega^2\rho^2 = \textit{ constant}$$

with $s = r - r_0$ and $\Omega = \frac{q}{m}\left(\frac{\partial A}{\partial r}\right)_{r=r_0}$, where r_0 is defined by $qA(r_0) = p_z$. Find the constant ρ and write down the general solution for $s(t)$. [3]

CORRIGENDUM: Corrected version of Question 2

Section S7 CLASSICAL MECHANICS

2. Consider a mass m attached to a ring of radius a by two identical springs of unstretched length l_0 ($l_0 < a$) and spring constant k (see diagram) – the springs are attached to points on opposite sides of the ring *i.e.* a distance $2a$ apart. The ring lies in $x-y$ plane and rotates at angular velocity Ω about the z axis. The mass is free to move in the $x-y$ plane.



(a) Show that the Lagrangian for the mass in terms of the rotating coordinates x_1 and x_2 (see diagram) is [8]

$$\mathcal{L} = \frac{1}{2}m\left[(\dot{x}_1 - \Omega x_2)^2 + (\dot{x}_2 + \Omega x_1)^2\right] - \frac{1}{2}k\left[(\sqrt{(a-x_2)^2 + x_1^2} - l_0)^2 + (\sqrt{(a+x_2)^2 + x_1^2} - l_0)^2\right].$$

[*Hint: recall that the kinetic energy in a rotating frame is* $\frac{1}{2}m(\mathbf{v}_{RF} + \mathbf{\Omega} \times \mathbf{r}_{RF})^2$, *where* $\mathbf{v}_{RF}$ *and* $\mathbf{r}_{RF}$ *are the velocity and position measured in the rotating frame.*]

(b) For small displacement from the centre of the ring we may write an approximate Lagrangian as:

$$\mathcal{L}' = \frac{1}{2}m\left[(\dot{x}_1 - \Omega x_2)^2 + (\dot{x}_2 + \Omega x_1)^2\right] - kx_2^2 - k'x_1^2$$

where $k' = k(a - l_0)/a$ [*you don't have to derive this equation.*] Obtain the Euler-Lagrange equations and identify the Coriolis and Centrifugal terms. [6]

(c) Take solutions of the form $x_1 = \bar{x}_1 e^{-i\omega t}$, $x_2 = \bar{x}_2 e^{-i\omega t}$ where $\bar{x}_1$ and $\bar{x}_2$ are constants and show that ω is a solution of:

$$(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2) = 4\Omega^2\omega^2$$

where $\omega_1^2 = (2k'/m - \Omega^2)$ and $\omega_2^2 = (2k/m - \Omega^2)$. [6]

(d) Show that when $\omega_1^2 > 0$ and $\omega_2^2 > 0$ then there are two positive solutions for ω^2. What happens when $\omega_1^2 < 0$ and $\omega_2^2 > 0$? Interpret this physically. [5]

3. Consider a Hamiltonian flow in phase space.

(a) Show that the flow in phase space is incompressible [*Liouville's theorem*]. [5]

(b) Define the poisson bracket $[a, b]$ and show that for a function $g(\mathbf{q}, \mathbf{p}, t)$

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \frac{\partial g}{\partial t} + [H, g]$$

hence show that if $\frac{\partial H}{\partial t} = 0$ then $\frac{\mathrm{d}H}{\mathrm{d}t} = 0$. [7]

(c) Consider the probability density of particles in phase space $f(\mathbf{q}, \mathbf{p}, t)$ – show that f is constant along trajectories. [5]

(d) Show that if $\frac{\partial H}{\partial t} = 0$ then $f = f(H)$ is a solution of the equation for f. [3]

(e) An axisymmetric system has $\frac{\partial \mathcal{L}}{\partial \theta} = 0$ and $\frac{\partial \mathcal{L}}{\partial t} = 0$. Write down the form of a time independent solution for f that is more general than the solution in part (d). [5]

Section S12 INTRODUCTION TO BIOLOGICAL PHYSICS

1. Describe in detail the structures of DNA and RNA that are commonly found in living cells and how these relate to their biological functions and to the evolution of species. [25]

2. Derive the Einstein relation between diffusion (D) and viscous drag (γ) coefficients of a polystyrene microsphere $1\mu m$ in diameter, suspended in water, by considering diffusion in a time-invariant 1-dimensional potential. [7]

Diffusion of the above microsphere may be modelled as a 3-dimensional random walk with discrete steps that occur at a rate k. If the mean-square step size is Δ obtain expressions for the mean-square displacement after m steps, and for the diffusion coefficient D, in terms of m, k and Δ. [*The mean-square displacement in a 1-dimensional walk after time t is $\langle x^2 \rangle = 2Dt$.*] [8]

The microsphere is held in a pair of optical traps, separated by $1\mu m$, and is observed to make sudden jumps between the traps at random times. On average, it spends twice as long in trap 1 as in trap 2. Draw a labeled sketch of the energy of the microsphere versus the position along the axis between the traps, and write an expression, in terms of the free energies G_1 and G_2 of the trapped states, for the ratio k_{12}/k_{21}, where k_{ij} are rate-constants for jumps from trap i to trap j. What is the numerical value of k_{12}/k_{21}? [5]

A constant flow of water is introduced in the direction from trap 1 to trap 2. What flow velocity is required to make the microsphere spend equal time in both traps? [5]

[*Stokes Law for the microsphere gives $\gamma = 6\pi\eta a$ where $\eta = 10^{-3}$ kg m^{-1} s^{-1} is the viscosity of water and a the radius of the sphere.*]

3. Describe the main reservoirs of free energy in living cells and the main flows between them. [8]

A bacterial cytoplasmic membrane has a thickness of $5nm$, a relative dielectric constant of 2.5, and surrounds a cell which is a cylinder $2\mu m$ long by $1\mu m$ wide with additional hemispherical end-caps. The net charge imbalance across the membrane, inside minus outside, is -40,000 times the proton charge and the pH is 7.0 outside and 7.5 inside the cell. What is the protonmotive force? [7]

Describe in detail how ATP is generated in living cells using the protonmotive force. [10]

Section S14 HISTORY OF PHYSICS

1. How did the use of instruments in natural philosophy transform knowledge of the natural world? [25]

2. What was novel about Newton's physical theories? [25]

3. To what extent did Carnot's depiction of an 'ideal' engine rely on practical experience with real engines? [25]

4. How did William Thomson harmonize the work of Rumford, Carnot, Joule and others to create the laws of thermodynamics? [25]

5. What was characteristic about 'Big Science'? [25]

Section S16 PLASMA PHYSICS

1. What is the physical significance of the Debye Length λ_{D}? Show that in a plasma with electron number density n_{e} and temperature T

$$\lambda_{\mathrm{D}} = \sqrt{\frac{\varepsilon_0 k_{\mathrm{B}} T}{n_{\mathrm{e}} e^2}}. \qquad [4]$$

What is meant by the terms *plasma parameter* and a *good plasma*? Many plasma phenomena are considered to be *collisionless*. What does this mean? Explain what is meant by a collision in this context. Write down an approximate relation between the collision time t_{c}, the plasma frequency ω_{p}, and the plasma parameter. [8]

The compressions and rarefactions that occur within one-dimensional plasma waves are assumed to be adiabatic, i.e. PV^{Γ} remains constant, and this gives rise to plasma oscillations in a warm plasma that obey the Bohm-Gross dispersion relation $\omega^2 = \omega_{\mathrm{p}}^2 + \Gamma k^2 v_{th}^2$, where $v_{th}^2 = k_{\mathrm{B}} T / m_e$. Explain why $\Gamma = 3$ for plasma waves in a *good* plasma, whereas a value of $\Gamma = 5/3$ is used in the derivation of the dispersion relation for adiabatic sound waves within an un-ionized monatomic gas. [4]

A sample of ${}^{9}_{4}\mathrm{Be}$ of density 1.85 g cm^{-3} is rapidly heated at constant volume by short pulse lasers such that it is doubly ionized. Deduce the plasma frequency of the system, ω_{p}, in units of eV. X-rays of energy 2962 eV are incident on the plasma exciting waves within it. Those x-rays that are inelastically scattered through an angle of $\pi/4$ radians are detected on a spectrometer, and a peak in the spectrum is observed at a photon energy of 2932 eV. Deduce the values of ω and k of the plasma waves that give rise to the inelastic x-ray scattering, and hence, assuming the Bohm-Gross dispersion relation is valid, calculate the temperature (in units of eV) of the solid density plasma. [6]

Determine the plasma parameter for this system, and hence comment on the potential validity or otherwise of your result. [3]

2. The Saha equation expresses the ratio of the number density of free-electrons, $n_{\rm e}$, to the number density of neutral hydrogen atoms, $n_{\rm H}$, when a hydrogen plasma is heated. It is given by

$$\frac{n_{\rm e}^2}{n_{\rm H}} = \frac{(2\pi m_{\rm e} k_{\rm B} T)^{3/2}}{h^3} \, \exp\left(-R/k_{\rm B}T\right),$$

where R is the ionization potential of atomic hydrogen.

Consider a cold system of un-ionized hydrogen atoms with number density n_0. When this system is heated to a temperature T' half of the atoms are found to be ionized. The number density of hydrogen atoms in a second cold system is $n_0 + \Delta n_0$. Show that this second system is now half ionized at a temperature of $T' + \Delta T'$, where

$$\frac{\Delta n_0}{n_0} \approx \frac{\Delta T'}{T'} \left(\frac{3}{2} + \frac{R}{k_{\rm B}T'}\right). \qquad [8]$$

For a fractional increase in n_0 of 0.1 it is found that the fractional increase in T' is 0.01. Give a physical reason for the increase in T' with an increase in the number density. Determine n_0 and T', and comment on the value of $R/k_{\rm B}T'$. [9]

In the derivation of the Saha equation, it is usual to ignore the excited electronic states of the hydrogen atom. Why is this usually justified? Illustrate your answer by considering the relative population of the $n = 2$ quantum states of atomic hydrogen in the above plasma at the temperature T'. [8]

3. Show that when a high power optical laser with intensity I and wavelength λ ablates material from a target the ablation pressure produced is given by

$$P_{\mathrm{A}} \approx C(I/\lambda)^{2/3}\,,$$

where C is a constant. Obtain an expression for C, and estimate the ablation pressure in Mbar for a laser with wavelength 0.35-μm operating at an intensity of 10^{15} Wcm^{-2}. [*You may assume that the ablation velocity of the plasma at the critical density surface is Mach 1.*] [8]

The Lawson criterion for energy gain in a Deuterium-Tritium (D-T) fusion reaction states that the product of the number density of ions , n_{i}, and the time for which they are confined, τ, is given by $n_{\mathrm{i}}\tau > 10^{20}\mathrm{m}^{-3}\mathrm{s}$. For fusion to occur the D-T fuel must be heated to temperatures of order 10 keV. Show that for inertial confinement fusion, where a sphere of D-T of radius r and density ρ must be heated, the Lawson criterion may be written as $\rho r > A$, and evaluate a rough value of A in units of g cm^{-2}. [6]

One concept to lower the threshold for inertial confinement fusion involves placing the fusion pellet in an initially uniform magnetic field. The magnetic field lines remain tied to the plasma during the laser-driven implosion, such that when the capsule implodes from an initial radius r_i, to final radius r_f, the magnetic field is increased by the square of the convergence ratio, i.e. $(r_i/r_f)^2$. The threshold for ignition is reduced because some of the fusion alpha particles (of energy 3.5 MeV) remain magnetically confined within the compressed fusion core owing to their Larmor radius being less than r_f, such that they deposit their energy within the fusion fuel, rapidly heating it further. For a capsule of initial diameter 1 mm, and for an implosion with a convergence ratio of 15, make an estimate of the initial magnetic field required for this phenomenon to start to take effect. [7]

Estimate the magnetic pressure within the compressed fusion core. [4]

Section S18 ADVANCED QUANTUM MECHANICS

1. Using Fourier transform or any other method, show that the Green's function $G(x,x')$ obeying the equation $\hat{L}\,G(x,x') = \delta(x-x')$ and the boundary conditions $G(x,x') \to C\,\exp(ik|x|)$ for $|x| \to \infty$, where $\hat{L} = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - E$ is the one-dimensional Schrödinger operator for a free particle with $E > 0$, C is a constant and $k = \sqrt{2mE}/\hbar$, is given by $G(x,x') = \frac{\mathrm{i}m}{k\hbar^2}\exp(\mathrm{i}k|x-x'|)$. [5]

A non-relativistic quantum particle of mass m is incident from the left on the one-dimensional potential $U(x) \le 0$, where $U(x) \to 0$ for $|x| \to \infty$. Show that the wave function of a stationary scattering state of the particle satisfies the integral equation

$$\psi_s(x) = \mathrm{e}^{\mathrm{i}kx} - \frac{\mathrm{i}m}{k\hbar^2}\int\limits_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}k|x-x'|}U(x')\psi_s(x')\mathrm{d}x'\,. \qquad [3]$$

Considering the asymptotics of $\psi_s(x)$ at $x \to \pm\infty$, define the reflection and transmission coefficients T and R. Show that the reflection coefficient is given by

$$R = \frac{m^2}{k^2\hbar^4}\left|\int\limits_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}kx}U(x)\psi_s(x)\mathrm{d}x\right|^2\,,$$

where $\psi_s(x)$ is the solution of the integral equation above. [4]

Find the solution of the integral equation for the potential $U(x) = -\alpha\delta(x)$, $\alpha > 0$. Find the transmission and reflection coefficients T and R, and show that $R + T = 1$. [3]

Consider developing a perturbation theory for $\psi_s(x)$ in the form $\psi_s(x) = \psi_s^{(0)}(x) + \psi_s^{(1)}(x) + \ldots$, where $\psi_s^{(0)}(x) = \mathrm{e}^{\mathrm{i}kx}$ is the solution of the free equation. Find R to leading order in the perturbation theory and compute it for the potential $U(x) = -\alpha\delta(x)$. Compare with the exact solution. Explain the physical meaning of the approximation made. [3]

Now consider stationary states with $E < 0$. Show that the wave function $\psi(x)$ for such states obeying the boundary conditions $\psi(x) \to 0$ for $x \to \pm\infty$ satisfies the integral equation

$$\psi(x) = -\frac{m}{\kappa\hbar^2}\int\limits_{-\infty}^{\infty}\mathrm{e}^{-\kappa|x-x'|}U(x')\psi(x')\mathrm{d}x'\,, \qquad [4]$$

where $\kappa = \sqrt{-2mE}/\hbar$. Using the equation, find the energy levels in the potential $U(x) = -\alpha\delta(x)$, $\alpha > 0$. How are they related to the singularities of the transmission coefficient in the same potential? [3]

2. Consider a non-relativistic quantum particle of mass m whose wavefunction $\psi(x)$ satisfies the one-dimensional Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + U(x)\psi(x) = E\psi(x)\,,$$

where the potential $U(x) \to 0$ for $|x| \to \infty$. Find the Green's function $G(x, x')$ of the free Schrödinger operator $\hat{L}\,G(x, x') = \delta(x - x')$, where $\hat{L} = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - E$ with $E < 0$. [3]

Show that the Schrödinger equation for the particle with $E < 0$ can be written as an integral equation

$$\psi(x) = -\frac{m}{\kappa\hbar^2}\int_{-\infty}^{\infty} \mathrm{e}^{-\kappa|x-x'|}U(x')\psi(x')\mathrm{d}x'\,, \qquad [2]$$

where $\kappa = \sqrt{-2mE}/\hbar$.

Consider a potential of the form

$$U(x) = \begin{cases} -U_0, & |x| \le \frac{a}{2}\,, \\ 0, & |x| > \frac{a}{2}\,, \end{cases}$$

where $U_0 > 0$.

a) Find the even and odd parity wave functions corresponding to the stationary states with $E < 0$ (bound states) in this potential. [4]

b) Using the continuity conditions for the wave function $\psi(x)$ and its derivative $\psi'(x)$ at $x = \pm a/2$, show that the conditions determining the bound state energies in the potential $U(x)$ are given by the equations

$$\begin{cases} \sqrt{\frac{U_0}{|E|} - 1}\,\tan\left[\frac{\kappa a}{2}\sqrt{\frac{U_0}{|E|} - 1}\right] = 1 & \text{(even parity states)}\,, \\ \sqrt{\frac{U_0}{|E|} - 1}\,\cot\left[\frac{\kappa a}{2}\sqrt{\frac{U_0}{|E|} - 1}\right] = -1 & \text{(odd parity states)}\,. \end{cases} \qquad [4]$$

c) Now consider a non-relativistic quantum particle of mass m and wavenumber k incident from the left on the potential $U(x)$. Show that the transmission coefficient $T(E)$, where $E = \hbar^2k^2/2m > 0$, is given by

$$T(E) = \left[\cos^2\zeta a + \frac{(k^2 + \zeta^2)^2}{4k^2\zeta^2}\sin^2\zeta a\right]^{-1}\,,$$

where $\zeta^2 = 2m(E + U_0)/\hbar^2$. [4]

d) Show that the singularities of the transmission coefficient $T(E)$ for complex values of $k = \mathrm{i}\sqrt{2m|E|}/\hbar$ correspond to the bound states energies in the potential $U(x)$. [*Hint: Use the trigonometric identity* $\tan 2x = 2\tan x/(1 - \tan^2 x)$.] [4]

e) What is the maximum value of $T(E)$? At what energies is it attained? Sketch (qualitatively) the function $T(E)$ for $E \ge 0$. [2]

f) Sketch (qualitatively) the location of the singularities of the transmission coefficient $T(E)$ in the complex k plane and in the complex E plane. [2]

3. The Dirac equation in an external electromagnetic field $A^\mu = (\Phi, \mathbf{A})$ is

$$\left[\gamma^\mu \left(\hat{p}_\mu - \frac{e}{c} A_\mu\right) - mc\right] \psi = 0\,,$$

where $\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$ is the four-component Dirac spinor. The Minkowski metric is given by $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$, $\hat{p}_\mu = i\hbar\partial_\mu$, and the Dirac matrices are

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix},$$

where σ^k are Pauli matrices obeying $\sigma_i \sigma_k = \delta_{ik} + \mathrm{i}\epsilon_{ikl}\sigma_l$. Consider a relativistic particle with spin 1/2 and charge e moving in a constant magnetic field $\mathbf{B} = \mathrm{curl}\,\mathbf{A} = (0, 0, B)$. Choose the gauge $\mathbf{A} = (-By, 0, 0)$, write down the Dirac Hamiltonian explicitly, then justify and use the *ansatz*

$$\psi = \mathrm{e}^{-\mathrm{i}\frac{E}{\hbar}t + \mathrm{i}\frac{p_x}{\hbar}x + \mathrm{i}\frac{p_z}{\hbar}z} \begin{pmatrix} \varphi(y) \\ \chi(y) \end{pmatrix},$$

to write the Dirac equation as a system of coupled equations for the spinors φ and χ. [4]

Show that the spinor φ satisfies the equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - \frac{(p_x c + eBy)^2}{\hbar^2 c^2} + \frac{E^2 - m^2c^4}{\hbar^2 c^2} - \frac{p_z^2}{\hbar^2} + \frac{eB}{\hbar c}\sigma_3\right) \varphi(y) = 0\,.$$ [5]

Show that this equation is identical to the Schrödinger equation for a harmonic oscillator,

$$\left(-\frac{\hbar^2}{2M}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \frac{M\omega^2\xi^2}{2}\right) \Psi(\xi) = \varepsilon\Psi(\xi)\,,$$

and identify the parameters M and ε. [*Hint: change variable to* $\xi = (p_x c + eBy)/\hbar c$.] [5]

Using the spectrum of the harmonic oscillator, $\varepsilon = \hbar\omega(n + 1/2)$, and the property $\sigma_3\varphi = \pm\varphi$, show that the energy levels E_{n,p_z} of the particle in a constant magnetic field satisfy the equation ($\mu = |e|\hbar/2mc$):

$$E_{n,p_z}^2 = m^2c^4 + 2mc^2\left[\frac{p_z^2}{2m} + \mu B(2n+1) \pm \mu B\right], \qquad n = 0, 1, 2, ...$$ [4]

A spinless relativistic particle of mass m and charge e in an external constant magnetic field $\mathbf{B} = (0, 0, B)$ obeys the stationary Klein-Gordon equation

$$\left[c^2\left(-\mathrm{i}\hbar\boldsymbol{\nabla} - \frac{e}{c}\mathbf{A}\right)^2 + m^2c^4\right] \phi(x, y, z) = E^2\phi(x, y, z)\,.$$

By reducing this equation to the Schrödinger equation for the harmonic oscillator, show that the energy levels of the particle satisfy

$$E_{n,p_z}^2 = m^2c^4 + 2mc^2\left[\frac{p_z^2}{2m} + \mu B(2n+1)\right], \qquad n = 0, 1, 2, ...$$ [5]

Based on these results, can you guess a formula for the energy levels of a relativistic particle of spin s in a constant magnetic field? Discuss the instability occuring for a spin 1 particle in the field $eB > m^2c^4/\hbar c$ at $p_z = 0$. [2]

Section S19 UNIFYING PHYSICS OF ACCELERATORS, LASERS AND PLASMA

The following constants will be useful (in addition to those on the data sheet):

Electron rest energy	0.511 MeV
Proton rest energy	938 MeV
Classical electron radius $r_e = 1/(4\pi\varepsilon_0)e^2/(mc^2)$	2.81×10^{-15} m
$C_q = \frac{55}{32\sqrt{3}}\frac{\hbar}{mc}$	3.84×10^{-13} m

1. Beam optics.

(a) Consider a cell made of two thin lens quadrupoles of equal and opposite focal length f, separated by two drifts with length L_1 and L_2. Using the thin lens approximation for the quadrupoles, write down the transfer matrix M for the cell. [7]

(b) Consider a ring whose lattice is obtained by repeating the cell N times. For what values of the integrated gradients of the quadrupole is the ring stable? Describe how the phase advance varies as a function of the quadrupole strength. [9]

[*Hint: use the Twiss representation for the transfer matrix.*]

(c) Calculate the chromaticity of the cell. Can the cell be operated at zero chromaticity? [9]

2. Consider an electron storage ring operating at 3 GeV, with a lattice made of 24 Double-Bend Achromat (DBA) cells.

(a) Give an estimate for the minimum emittance achievable in such a ring (assume a damping partition number $J_x = 1$). If the magnetic field of the dipole is 1.4 T, calculate the critical energy of the bending magnets. [5]

(b) Assume we want to upgrade this lattice to a 48 DBA cells. Give an estimate for the minimum emittance achievable in such a ring. Assuming that the length of each dipole has not changed in the upgrade, calculate the new critical energy for the dipole radiation. Describe what is the change in the characteristics of the bending magnet radiation (i.e. power, spectrum and angular distribution). [10]

(c) Describe the optics design strategies for minimising the equilibrium emittance. Sketch a few possible solutions based on achromatic and non-achromatic cells. [10]

3. A short laser pulse with peak power of 40 TW is focused into plasma with density $n = 4 \times 10^{18}$ cm^{-3} so that the laser power density at the focus is approximately 10^{18} W cm^{-2}.

(a) The plasma is underdense for this laser and the ratio of the critical density n_c to the plasma density is $n/n_c \approx 0.002$. Estimate the laser wavelength λ. [5]

(b) Estimate the plasma frequency, ω_p, plasma wavelength, λ_p, and the maximum accelerating gradient, eE_{max}, in the nonlinear plasma wave created by the laser pulse. [5]

(c) Estimate the maximum length of the laser pulse σ_{max} (in μm and fs) which would allow the plasma to reach the maximum accelerating gradient. [5]

(d) Estimate the maximum amplitude of the normalized vector potential, a_0, and determine if the motion of plasma electrons is weakly or highly relativistic. [5]

(e) Estimate the maximum possible energy gain in one accelerating stage, assuming that it is limited by dephasing between the laser pulse and accelerating electrons. [5]

Section S25 CLIMATE PHYSICS

1. Diffuse upward irradiance I at an individual wavelength in an atmosphere containing only a single radiatively active gas is governed by Schwarzschild's equation:

$$-\frac{\mathrm{d}I}{\mathrm{d}\tau} + I = \pi B(\tau) \text{ where } \mathrm{d}\tau = -k\rho\mathrm{d}z \ ,$$

k is the diffuse absorption coefficient, ρ the absorber density, and $B(\tau)$ the Planck function at that wavelength, given the temperature at optical depth τ. With reference to Beer-Lambert's and Kirchhoff's laws, explain the physical significance of all terms and signs, and describe how you would expect I to vary with optical depth τ in the case of (a) negligible atmospheric emission above a warm surface and (b) an infinitely thick isothermal atmosphere. [8]

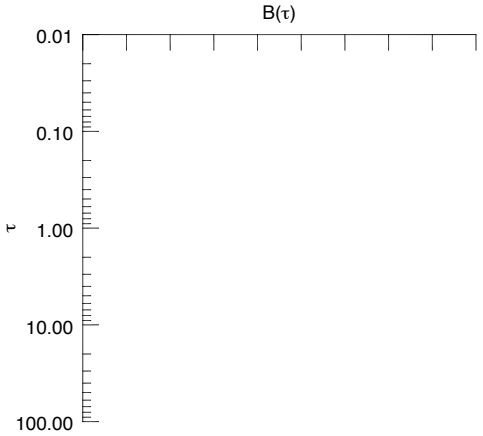
Starting from Schwarzschild's equation and using an integrating factor of $\mathrm{e}^{-\tau}$, or otherwise, show that upward irradiance at the top of an infinitely thick non-isothermal atmosphere is given by

$$I_0 = \pi \int_0^\infty B(\tau)\mathrm{e}^{-\tau}\mathrm{d}\tau \ ,$$

and provide a physical interpretation of this expression. [5]

Starting from the hydrostatic approximation and assuming $k \propto p$ and weak vertical temperature gradients, show that τ declines approximately exponentially with height and derive an expression for the optical depth scale height, H_τ in terms of the specific gas constant for air R_a and the ambient temperature T. [6]

Assuming that temperatures decline with height at a constant lapse rate Γ up to a tropopause above which they are constant, and B is an approximately linear function of temperature over the relevant range, copy the axes below and sketch $B(\tau)$ corresponding to a wavelength for which the optical depth of the tropopause lies between 0.1 and 1. On the same axes, add clearly labelled lines to show $B(\tau)$ following an instantaneous doubling and halving of the mixing ratio of the radiatively active gas, assuming no change in atmospheric temperatures. [6]



2. For small changes in height, the temperature of a unit mass parcel of moist rising air is approximately given by $c_{\mathrm{p}}\mathrm{d}T = V\mathrm{d}p - L\mathrm{d}q$ where c_{p} is the specific heat capacity at constant pressure, L is the latent heat of vaporisation of water, $\mathrm{d}q$ is the change in specific humidity due to condensation and all other symbols have their conventional meaning. Derive an expression for the lapse rate Γ_{d} for dry ascent ($\mathrm{d}q = 0$) and show that condensation acts to generate a smaller lapse rate than Γ_{d} in the Earth's atmosphere. Explain further why the lapse rate is expected to fall as the atmosphere warms. [9]

Explain the concept of a planet's "effective emission temperature", T_{e}, and derive an expression for the Earth's T_{e} in terms of the solar constant S_0 and planetary albedo α. An instantaneous doubling of atmospheric CO_2 concentrations reduces outgoing infrared radiation by $3.7\,\mathrm{W\,m^{-2}}$. Calculate the resulting equilibrium surface and tropospheric warming, assuming the Earth is initially in radiative equilibrium, $\alpha = 0.3$ and no properties of the atmosphere other than temperature are allowed to change. [5]

The increase in specific humidity due to tropospheric warming reduces outgoing radiation by $2\,\mathrm{W\,m^{-2}\,K^{-1}}$, while the declining lapse rate increases outgoing radiation by $0.5\,\mathrm{W\,m^{-2}\,K^{-1}}$. Explain the sign of the effect of the changing lapse rate. Calculate the equilibrium warming caused by a doubling of CO_2 concentrations accounting for these composition and lapse rate changes, assuming no change in planetary albedo. [3]

In addition to the above composition and lapse-rate changes, cloud short-wave reflectivity decreases with warming temperatures in a climate model, altering the albedo such that $\alpha = a_0 - a_1 T - a_2 T^2$, where T is the increase in global average surface temperature above its pre-industrial equilibrium value, $a_0 = 0.3$, $a_1 = 0.003\,\mathrm{K^{-1}}$ and $a_2 = 0.0002\,\mathrm{K^{-2}}$. Calculate the equilibrium warming caused by a doubling of CO_2 concentrations accounting for these composition, lapse rate and albedo changes, assuming these cloud changes have no impact on infrared radiation. If pre-industrial CO_2 concentrations is 280 ppm, find the maximum CO_2 concentration for which the climate of this model has a stable climate. [8]

3. The excess mass of CO_2 in the atmosphere, $C(t)$, resulting from fossil CO_2 emissions, $E(t)$, may be approximated by the following simple model:

$$\begin{aligned} \frac{\mathrm{d}C}{\mathrm{d}t} &= E - k(C - C_1) \\ \frac{\mathrm{d}C_1}{\mathrm{d}t} &= k(C - C_1) \ . \end{aligned}$$

Provide a physical interpretation of C_1 and find an expression for $C(t)$ resulting from a sudden injection of one unit mass of CO_2 into an unperturbed atmosphere at $t = 0$ assuming k may be treated as constant. Sketch your result, clearly labelling any time-constants. [6]

The global average surface temperature response $T(t)$ to emissions $E(t)$ may be approximated by

$$\frac{\mathrm{d}T}{\mathrm{d}t} = b_0 C - b_1 T \ .$$

State and briefly explain how the value of b_0 depends on background atmospheric CO_2 concentrations: you do not have to give numerical values. Find expressions for (a) $\mathrm{d}T/\mathrm{d}t$ immediately after a unit mass emission of CO_2 and (b) for the asymptotic value of $T(t)$ as $t \gg k^{-1}$ and $t \gg b_1^{-1}$. Hence, or otherwise, sketch the evolution of $T(t)$ in response to a unit mass emission of CO_2 assuming $0 \ll k \ll b_1$, clearly labelling any time-constants. [10]

The Social Cost of Carbon Dioxide (SC-CO_2) is given by

$$\text{SC-}CO_2 = \int_{t=0}^{\infty} D'(t) T(t) \mathrm{e}^{-rt} \mathrm{d}t$$

where $T(t)$ is the response to a unit mass emission of CO_2 occurring at $t = 0$, r is the discount rate and $D'(t)$ is the rate of increase of the total cost per year of climate-change-related impacts per degree of warming at time t. Find an expression for the SC-CO_2 assuming that $r \ll b_1$ and that D' is constant over the relevant adjustment timescales. [6]

State, with your reasoning, how would you expect $D'(t)$ to change as the world warms, assuming economic activities are optimally adapted to 20^{th} century climate conditions. [3]

Section S27 PHILOSOPHY OF SPECIAL RELATIVITY

1. What is the role, if any, of the relativity of simultaneity in solving the twin paradox? [25]

2. In what sense, if any, does special relativity imply a 'tenseless' conception of time? [25]

3. Either: Are the Lorentz transformations empirically testable? Or: Derive the Lorentz transformations. What role, if any, does the behaviour of light play in your derivation? [25]

4. In what sense, if any, is Newton's first law empirically testable? [25]

Section S28 PHILOSOPHY OF QUANTUM MECHANICS

1. What is Bell's theorem, and how does it bear on the EPR paradox? [25]

2. Can the Everett interpretation account for indeterminism – or at least the appearance of indeterminism? [25]

3. In terms of what the world is fundamentally like, does the de Broglie-Bohm pilot-wave theory mark a significant departure from classical mechanics? [25]

4. Is quantum mechanics in need of an interpretation? [25]

5. Does a dynamical collapse theory require additional posits – for example, concerning 'mass density'? [25]