A10289W1

FIRST PUBLIC EXAMINATION

Long Vacation

Preliminary Examination in Physics

Paper CP4: MATHEMATICAL METHODS 2

also

Preliminary Examination in Physics and Philosophy

Thursday 7 September 2017, 9.30 am – 12:00 pm

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a new page.

The use of approved calculators is permitted.

A list of physical constants, mathematical formulae and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. (a) For the case where a is a constant and x is a position variable, use the Taylor series expansion for $\cos(a+x)$ to show that

$$\cos(a+x) = \cos a \cos x - \sin a \sin x. \quad [2]$$

(b) Express the hyperbolic cotangent function, $\coth x$, in terms of exponential functions of x where x is real. Find the Maclaurin series expansion for $\coth x$ up to and including third order in x by using the Maclaurin series expansion for the exponential function. [3]

2. Using the transformation $u = y - x$ and $v = y + x$, evaluate the double integral

$$\iint_R e^{(y-x)/(y+x)} \, \mathrm{d}x\mathrm{d}y$$

over the triangular region R in the xy plane with vertices $(0,0)$, $(3,0)$, and $(0,3)$. [7]

3. (a) Sketch the vector field $\mathbf{v} = v_0 e^{-y^2}\mathbf{i}$ where v_0 is a positive constant. Calculate the curl of $\mathbf{v}$ and explain the physical significance of the result. [2]

(b) Find the divergence and curl of $\mathbf{r} \times \mathbf{a}$ where $\mathbf{r}$ is the position vector and $\mathbf{a}$ is a constant vector. [4]

4. (a) Given a function $f(x,y,z)$, show that at any point where ∇f is well defined and non-zero, ∇f is perpendicular to the surface $f(x,y,z) = C$ (where C is a constant) passing through that point. [2]

(b) Find the surface area of the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the xy plane. [4]

5. What is a *line integral* ? Evaluate $\int_C \mathbf{a} \cdot \mathrm{d}\mathbf{r}$ where $\mathbf{a} = (x-y)\mathbf{i} + (x+y)\mathbf{j}$ along the parabola $y^2 = x$ from $x = 0$ to $x = 1$. [4]

6. Find the positions and nature of the stationary points of the function

$$f(x,y) = x^2 + 2y^2 - x^2 y.$$

Sketch the contours of constant f. [7]

7. Determine whether $\mathbf{F} = 2x\mathbf{i} - 6y\mathbf{j} + 8z\mathbf{k}$ is conservative. Evaluate the line integral

$$\int_C \mathbf{F} \cdot \mathbf{dr}$$

along the curve C represented by the parametric equations

$$\begin{aligned} x(t) &= (2 + \cos(3t)) \cos t \\ y(t) &= (2 + \cos(3t)) \sin t \\ z(t) &= \sin(3t) \end{aligned}$$

from $t = 0$ to $t = \pi/2$. [5]

Section B

8. (a) Prove the identity

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$$

where $\mathbf{F}$ and $\mathbf{G}$ are vector fields. [5]

(b) Show that

$$\iiint_V \nabla \times \mathbf{G} \, \mathrm{d}V = \iint_S \hat{\mathbf{n}} \times \mathbf{G} \, \mathrm{d}S$$

where $\mathbf{G}$ is a vector field and $\hat{\mathbf{n}}$ is a unit vector normal to the surface S bounding the closed volume V. [Hint: Replace $\mathbf{G}$ in the divergence theorem with $\mathbf{G} \times \mathbf{C}$ where $\mathbf{C}$ is an arbitrary constant vector.] [5]

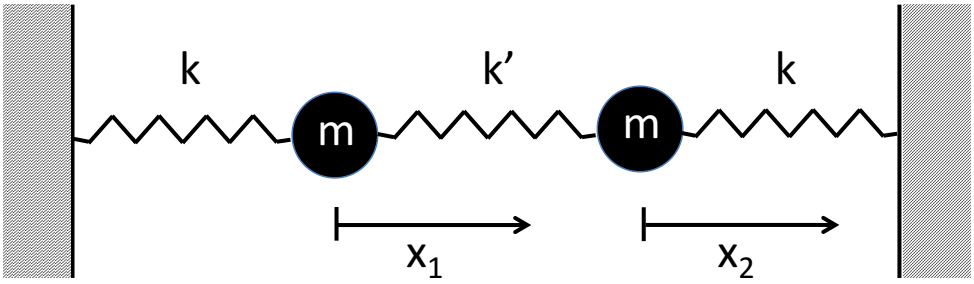
(c) If $\mathbf{F} = (\mathbf{b} \cdot \mathbf{r})\mathbf{b}$ where $\mathbf{b}$ is a constant vector, $\mathbf{r}$ is the position vector, and S is a closed surface, show that

$$\iint_S \mathbf{F} \times \mathrm{d}\mathbf{S} = 0.$$

[5]

(d) Verify the result obtained in part (c) by performing the surface integral for the special case where $b = (0, 1, 0)$ and S is the bounding surface of the hemisphere with radius $r \leq 1$ and $y \geq 0$. [5]

9.



Two equal masses, m, are connected to each other by a massless spring with spring constant k'. Each of the masses is also connected to a wall by a massless spring with constant k as shown in the figure above. The two masses are constrained to move along the x axis, free from friction and subject only to the forces imparted by the springs along the x axis. Assume that $k' = \epsilon k$ where ϵ is a positive constant and that in equilibrium, the springs are not stretched. x_1 and x_2 give the displacements of the masses away from their respective equilibrium positions.

(a) Write the equations of motion for the system. [4]

(b) Find its normal mode frequencies and amplitudes. [6]

(c) At time $t = 0$, the left mass is positioned at $x_1 = d$ and the right mass is at $x_2 = 0$. Knowing that the two masses are released from rest, find expressions for the subsequent time evolution of the displacements of both masses. [5]

(d) Assuming weak coupling between the two masses ($\epsilon \ll 1$), demonstrate that beats are generated. Find the time at which the second mass first comes to rest after the oscillations have commenced. [5]

10. (a) By performing the change of variable $x = e^t$, express the equation

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 3x \frac{\mathrm{d}y}{\mathrm{d}x} - 7y = 0$$

as a second order differential equation in t with constant coefficients. [7]

(b) When F is expressed as a function of x and t, the wave equation is given by

$$\frac{\partial^2 F}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 F}{\partial t^2} = 0$$

where v is a constant. Perform the change of variables $r = x + vt$, $s = x - vt$ to show that the wave equation given above is equivalent to

$$\frac{\partial^2 F}{\partial r \partial s} = 0.$$

Solve the latter equation to obtain d'Alembert's solution to the wave equation. [8]

(c) The function $z(x, y)$ satisfies the equation

$$\frac{\partial^2 z}{\partial x \partial y} = x^2 y.$$

Find $z(x, y)$ given that $z(x, 0) = x^2 - 1$ and $z(1, y) = \sin y$. [5]

11. (a) Show that two waves of equal amplitude travelling in opposite directions combine to form a standing wave. Discuss the difference between a travelling wave and a standing wave. [4]

(b) Consider a uniform string of length L and tension T with one end at $x = 0$ and the other end at $x = L$. Give expressions for small transverse displacements $y(x, t)$ of the string and determine the set of possible wavenumbers for the cases where:

(i) both ends of the strings are fixed;

(ii) the $x = 0$ end is fixed, but the $x = L$ end is free to oscillate in the $\pm y$ direction;

(iii) both ends are free to oscillate in the $\pm y$ direction.

Sketch the three lowest frequency modes of oscillation for each of the above three cases. [6]

(c) A string of linear mass density μ is joined at $x = L$ to a string with linear mass density $\mu/9$ and length $3L$ to form a composite string of total length $4L$. The end of this composite string is fixed at $x = 0$ and its end at $x = 4L$ is free to oscillate in the $\pm y$ direction. Show that for the string to be set into vibration with a standing wave pattern, the wavenumber k_1 in the denser part of the string must obey the following condition

$$\sin(k_1 L) = \pm\frac{\sqrt{3}}{2}.$$ [10]