A10289W1

FIRST PUBLIC EXAMINATION

Trinity Term

Preliminary Examination in Physics

Paper CP4: MATHEMATICAL METHODS 2

also

Preliminary Examination in Physics and Philosophy

Thursday 8 June 2017, 2.30 pm – 5 pm

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a new page.

The use of approved calculators is permitted.

A list of physical constants, mathematical formulae and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. Find the value of the constant b that makes the following

$$(y\sin x + b\sin y)\mathrm{d}x + (-\cos x + y^2 + 3x\cos y)\mathrm{d}y$$

the differential of a function $f(x, y)$. For this value of b, find the function $f(x, y)$ that satisfies the condition $f(0, 2) = 0$. [6]

2. Find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 4$$

by transforming to a coordinate system (u, v, w) where the ellipsoid is a sphere ($u^2 + v^2 + w^2 = 4$) and finding the Jacobian to perform the volume integration in the new coordinates. Assume a, b, and c are constants. [5]

3. (a) Estimate $\sqrt{26}$ using a Taylor expansion out to second order. [3]

(b) Use a Maclaurin series to evaluate

$$\lim_{x\to 0}((1+x)^{1/x}).$$ [3]

4. State *Stokes' theorem* for the vector field $\mathbf{A}$. Given

$$\mathbf{A}(x, y) = -\frac{y}{2}\mathbf{i} + \frac{x}{2}\mathbf{j},$$

calculate the area of the region enclosed by the curve C given by $x = \sin(2t)$ and $y = \sin t$ where t is a parameter in the range $0 \le t \le \pi$. [7]

5. Using the fact that integration is the inverse of differentiation, find $\partial F/\partial x$ and $\partial F/\partial y$ for the case where

$$F(x, y) = \int_0^{xy^2} e^{t^2}\mathrm{d}t.$$ [5]

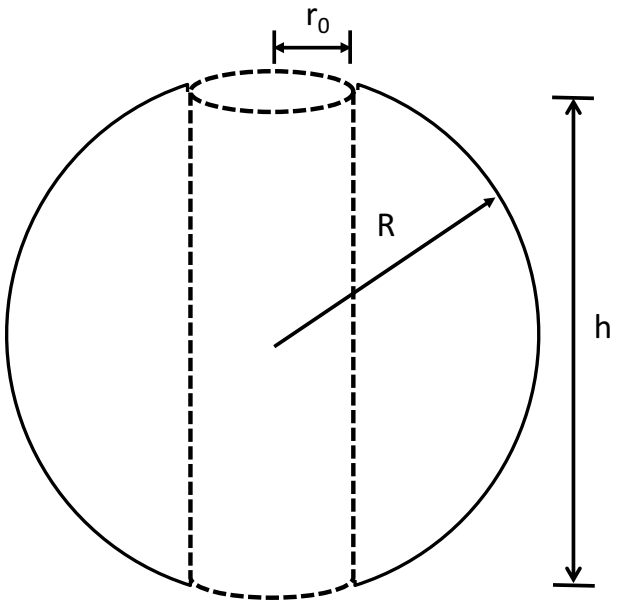
6. Define what is meant by a *standing wave*. A string of length L and tension T is fixed at $x = 0$ but at $x = L$ it is free to oscillate in the $\pm y$ direction. Assuming small displacements, give expressions for the displacement amplitudes $y(x, t)$ of the string and determine the set of possible wave numbers. [6]

7. The function w is continuous and differentiable and can be expressed either in terms of (u, v) or in terms of (x, y) where $u = x + y$ and $v = x - y$. Show that

$$\left(\frac{\partial w}{\partial x}\right)_y \left(\frac{\partial w}{\partial y}\right)_x = \left(\left(\frac{\partial w}{\partial u}\right)_v\right)^2 - \left(\left(\frac{\partial w}{\partial v}\right)_u\right)^2.$$ [5]

Section B

8.



A cylindrical hole of height h and radius r_0 is drilled all the way through a sphere of radius R. The axis of the cylindrical hole passes through the center of the sphere and $h < 2R$.

(a) By integrating, find the volume of the remaining solid in terms of h. Your answer should be independent of the radius of the sphere and the radius of the cylinder. [6]

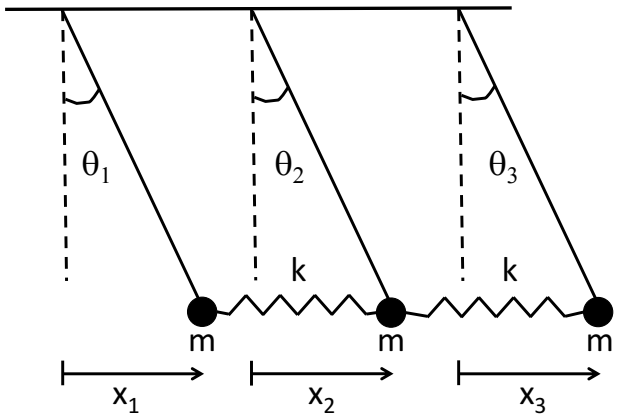
(b) State the *divergence theorem* and use it to show that a volume V is given by the following integral

$$V = \frac{1}{3}\int\int_S \hat{\mathbf{n}} \cdot \mathbf{r}\, \mathrm{d}S$$

where S is the surface enclosing the volume V, $\hat{\mathbf{n}}$ is a unit vector normal to the surface S and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. [4]

(c) By performing the surface integral in (b), show that you recover the same expression for the volume as you found in (a). [10]

9. Three simple identical pendula each consisting of a massless string of length l and a bob of mass m are coupled by two springs of spring constant k. The pendula are all fixed to a horizontal support and swing in a single plane perpendicular to it as in the diagram below.



Ignore friction at the pivot points and dissipation due to air resistance. Displacements from each pendulum's equilibrium position are given by x_1, x_2 and x_3. Assume that each spring is in its unstretched position when all three masses are at their equilibrium positions. The equations of motion for this system are given by

$$-m\,\frac{\mathrm{d}^2x_1}{\mathrm{d}t^2} = a\,m\,x_1 + b(x_1 - x_2)$$

$$-m\,\frac{\mathrm{d}^2x_2}{\mathrm{d}t^2} = a\,m\,x_2 + b(x_2 - x_1) + b(x_2 - x_3)$$

$$-m\,\frac{\mathrm{d}^2x_3}{\mathrm{d}t^2} = a\,m\,x_3 + b(x_3 - x_2)$$

where a and b are positive constants.

(a) Explain the origin of the terms (including the signs) in the equations of motion and give expressions for a and b in terms of k, l and the gravitational acceleration g. [3]

(b) When a normal mode is excited, how are the frequencies and phases of the three masses related? [1]

(c) Show that the normal mode frequencies of the system are

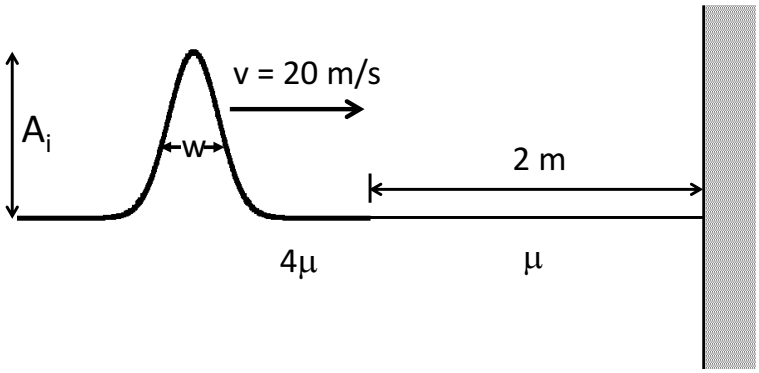
$$\omega_1 = \sqrt{\frac{g}{l}} \qquad \omega_2 = \sqrt{\frac{g}{l} + \frac{k}{m}} \qquad \omega_3 = \sqrt{\frac{g}{l} + \frac{3k}{m}}.$$ [5]

(d) Assuming zero initial velocities, explain how one can excite each normal mode of the system. [5]

(e) For the case where the three masses are released from rest with initial displacements from their equilibrium positions of $x_1 = 0$, $x_2 = \epsilon$ and $x_3 = 0$, find expressions for the subsequent displacement of each of the three masses as a function of time and demonstrate that beats are generated. [6]

10. For the case of a small amplitude travelling wave $y(x,t) = f(x - vt)$ moving with velocity v along a uniform string of linear mass density μ and tension T, derive expressions for the kinetic and potential energy of a short length, δx, of that string. Hence explain why the kinetic plus potential energy of a travelling wave on a string is proportional to the square of the amplitude of the wave. [5]

A 2 m long string of linear mass density μ is joined at one end to a heavier string of linear mass density 4μ, while its other end is fastened to a wall. The strings are under a constant tension. A single pulse with amplitude A_i and width w travels along the heavier string with velocity $v = 20$ m/s towards the lighter string as shown in the figure below.



The ratios of the reflected or transmitted amplitude (A_r, A_t) to the incident amplitude (A_i) of a wave are given by the following formulae where k_1 and k_2 are the wave numbers in strings of different linear mass density μ

$$\frac{A_r}{A_i} = \frac{k_1 - k_2}{k_1 + k_2} \qquad \frac{A_t}{A_i} = \frac{2k_1}{k_1 + k_2}.$$

(a) Rewrite the expressions for $\frac{A_r}{A_i}$ and $\frac{A_t}{A_i}$ given above in terms of phase velocities. Compute numerical values for $\frac{A_r}{A_i}$ and $\frac{A_t}{A_i}$ at the interface between the heavy and light string for the cases where (i) the pulse moves from the heavy to the light string (ii) the pulse moves from the light to the heavy string. [3]

(b) Through a series of sketches, show how the incident pulse is reflected back into the heavy string at the following instances

(i) some time after it first crosses from the heavy to the light string

(ii) when the incident pulse first reflects off the wall and

(iii) when it subsequently crosses from the light to the heavy string.

For each of these instances, indicate how the pulse shape (amplitude and width) changes from its original form. [5]

(c) By considering how the amplitude of the wave changes upon reflection, show that after an infinite number of reflections off the wall, all the energy of the incident pulse is reflected. [7]

11. Let $f(x, y)$ be a continuous and differentiable two dimensional function of variables x and y.

(a) By means of a sketch or otherwise, explain the meaning of

$$\frac{\partial^2 f}{\partial x \partial y}.$$

[3]

(b) What is a *stationary point*? From the two dimensional Taylor series, derive the conditions on $f(x, y)$ used to identify the nature of its stationary points. [6]

(c) Where does the function $f(x, y) = \cos(\sqrt{(x^2 + y^2)})$ take on its lowest values? Are these coincident with the minima of the function? Justify your answer. [3]

(d) Find an equation in plane polar coordinates (i.e. r as a function of θ) for the contour passing through the saddle point of the function $f(x, y) = 4xy - x^4 - y^4$. Identify all other stationary points of this function and sketch contours of constant f, including the one passing through the saddle point. [8]