A10285W1

FIRST PUBLIC EXAMINATION

Long Vacation

Preliminary Examination in Physics

Paper CP3: MATHEMATICAL METHODS 1

also

Preliminary Examination in Physics and Philosophy

Wednesday 6^{th} September 2017, 9:30 am – 12:00 noon

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a new page.

The use of calculators is **not** *permitted.*

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. (a) Use de Moivre's theorem to derive an expression for $\cos 3\theta$ in terms of $\cos\theta$ and $\sin\theta$, where θ is a real variable.

(b) Find all the roots of the equation $z^3 = 1$, where z is a complex variable.

(c) Show that $\dfrac{1+\mathrm{e}^{-i\theta}}{1-\mathrm{e}^{-i\theta}} = -i\cot(\theta/2)$, where θ is a real variable. [7]

2. Find the general solution $y(x)$ for the differential equation

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 3xy = 1\ .$$

[5]

3. Find the general solution $y(x)$ for the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^2y^2\ln x.$$

[6]

4. Let $\mathbf{A}$ be an orthogonal matrix.

(i) Show that $\det\mathbf{A} = \pm 1$.

(ii) Assume that the orthogonal matrix $\mathbf{A}$ is 2×2 with complex elements. Express the matrix elements of $\mathbf{A}$ in terms of two complex variables a and b.

(iii) Now express a and b in terms of the hyperbolic functions $\cosh\theta$ and $\sinh\theta$ where θ is a real number. [8]

5. Consider the set of linear equations in $\mathbb{R}$ with variables x, y, z,

$$\begin{aligned} 5x + 2y + 3z &= 4 \\ x + 2y + z &= 2 \\ 3x + \alpha y + 2z &= \beta \end{aligned}$$

where α and β are real constants.

(a) For what values of α and β will the above equations have no solution and for what values of α and β will the equations have more than one solution?

(b) For $\alpha = \beta = 6$, solve the above equations for x, y, z using either Cramer's rule or the raw reduction (Gaussian elimination) method. [7]

6. The position vectors $\mathbf{r}_i$ $(i = 1, 2)$ of points on two lines in 3-D are given by $\mathbf{r}_i = \mathbf{a}_i + \mathbf{m}_i t_i$, where $\mathbf{a}_i$ and $\mathbf{m}_i$ are constant vectors and t_i is a scalar parameter.
Derive an expression for the shortest distance between the two lines. [3]

Write down the vector equations of the two lines

$$\frac{(x-2)}{2} = (y-3) = \frac{(z+1)}{2} \quad \text{and} \quad (x+2) = \frac{(y+1)}{2} = (z-1)\ .$$

Find the shortest distance between the lines. [4]

Section B

7. (a) A particle of mass m is attached to one end of a spring of force constant k and executes simple harmonic oscillations. The particle's displacement from equilibrium x as a function of time t satisfies the differential equation,

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + {\omega_0}^2 x = 0 \ .$$

Find an expression for the displacement x as a function of time that satisfies the initial conditions:

$$\text{at } t = 0,\ x = x_0,\ \frac{\mathrm{d}x}{\mathrm{d}t} = 0.$$

Find an expression for ω_0 and write expressions for the kinetic energy and the potential energy of the spring as a function of time. [4]

(b) Assume now that damping is introduced into the system which can be expressed by adding the term $2\gamma\dfrac{\mathrm{d}x}{\mathrm{d}t}$, where γ is the damping constant, to the left-hand side of the differential equation given in part (a).
Derive general expressions (with two constants A, B) for the displacement $x(t)$ for the following two cases:

(i) $\gamma < \omega_0$

(ii) $\gamma = \omega_0$. [8]

(c) For the underdamped case, derive an expression for the total energy of the oscillator as a function of time assuming that $\gamma << \omega_0$ and the same initial conditions given above (at $t = 0,\ x = x_0,\ \dfrac{\mathrm{d}x}{\mathrm{d}t} = 0$). Find the time that it takes the oscillator to loose half of its initial energy. [4]

(d) Assume now that damping is removed ($\gamma = 0$) and the oscillator is driven by an external force of the form $F = F_0 \cos(\omega t - \phi)$, where F_0 is the force amplitude and ϕ is a phase constant. Find an expression for the displacement x as a function of time. Comment on the cases $\omega \gg \omega_0$ and $\omega = \omega_0$. [4]

8.

(a) Obtain and carefully sketch the locus in the complex plane defined by $\mathrm{Re}(z^{-1}) = 1$. On the same diagram, sketch the locus defined by $\mathrm{Im}(z^{-1}) = 1$. At what angle do these loci intersect one another? Show that the unit circle centred at the origin of the complex plane, touches both loci but crosses neither of them. [6]

(b) A plane in $\mathbb{R}^3$ is defined as $2x + 3y + 6z = 30$. what is the shortest distance from the plane to the point (x_0, y_0, z_0)? [5]

(c) Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors in $\mathbb{R}^3$.

(i) Prove that $|\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$. [4]

(ii) Show that the above relation may be written in the form

$$|\mathbf{a} \times \mathbf{b}|^2 = \mathbf{a}^T \mathbf{M} \mathbf{a} \, ,$$

where $\mathbf{M}$ is a 3×3 matrix and find the elements of the matrix $\mathbf{M}$. For the case $\mathbf{b} = (b_1, b_2, 0)$, calculate the determinant of $\mathbf{M}$. [5]

9. (a) Let V be a vector space over a field F. Define what is meant by the statement that the vectors $\mathbf{v_1}, \mathbf{v_2}, ...\mathbf{v_n} \in V$ are linearly independent.
For what values of the scalar α would the three vectors in $\mathbb{R}^3$

$$\begin{pmatrix} \alpha \\ 3 \\ 2 \end{pmatrix} , \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} , \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} ,$$

be linearly dependent? [4]

(b) Let V be a vector space over the field of complex numbers $\mathbb{C}$, with orthonormal basis set $\{\mathbf{e}_1, \mathbf{e}_2, ...\mathbf{e}_n\}$. Define the hermitian scalar product of two vectors $\mathbf{v}, \mathbf{w} \in V$ and show that the hermitian scalar product of the two vectors can be written as

$$\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i=1}^{n} \langle \mathbf{v}, \mathbf{e}_i \rangle \langle \mathbf{e}_i, \mathbf{w} \rangle \, .$$

[5]

(c) The linear map $P : \mathbb{R}^3 \to \mathbb{R}^3$ that represents reflection of a vector $\mathbf{v}$ from a plane that passes though the origin may be written as:

$$P(\mathbf{v}) = \mathbf{v} - 2(\mathbf{n} \cdot \mathbf{v})\mathbf{n}$$

where $\mathbf{n}$ is a unit vector perpendicular to the plane.
Show that the map is linear and verify that the angle of incidence (the angle that the incident vector makes with $\mathbf{n}$) is equal to the angle of reflection (the angle that the reflected vector makes with $\mathbf{n}$). [5]

Show that $P^2(\mathbf{v}) \equiv P(P(\mathbf{v})) = \mathbf{v}$, and hence find the eigenvalues of the map.
Find the matrix $\mathbf{A}$ that represents the map using the orthonormal basis set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}\}$. [6]

10.

(a) Let $\mathbf{A}$ and $\mathbf{B}$ be two invertible matrices.

(i) Show that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

(ii) Assuming that $\mathbf{A}$ is a 2×2 matrix with complex off-diagonal elements, give examples of matrix elements of $\mathbf{A}$ such that $\mathbf{A}$ is

(1) a symmetric matrix

(2) a hermitian matrix

(3) a unitary matrix. [7]

(b) The equation of a conical section with respect to an (x, y) coordinate system in $\mathbb{R}^2$ is given by:

$$f(x, y) = 2x^2 + 4xy - y^2 = 6 \, .$$

Find the symmetric matrix $\mathbf{M}$ that satisfies the relation

$$\begin{pmatrix} x & y \end{pmatrix} \mathbf{M} \begin{pmatrix} x \\ y \end{pmatrix} = f(x, y) \, .$$

[2]

Use matrix diagonalization to show that the above relation can be written as

$$\begin{pmatrix} x' & y' \end{pmatrix} \mathbf{D} \begin{pmatrix} x' \\ y' \end{pmatrix} = f(x', y') \, ,$$

where D is a diagonal matrix and (x', y') are the axes of a coordinate system which is rotated by an angle θ with respect to the (x, y) coordinate system. Find elements of the matrix D and give the value of $\sin\theta$. [7]

Hence show that $f(x, y) = 1$ is the equation of a hyperbola and find the slopes of its asymptotes. [4]