A10285W1

FIRST PUBLIC EXAMINATION

Trinity Term

Preliminary Examination in Physics

Paper CP3: MATHEMATICAL METHODS 1

also

Preliminary Examination in Physics and Philosophy

Wednesday 7 June 2017, 2.30 pm – 5 pm

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a new page.

The use of calculators is **not** *permitted.*

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. Evaluate the following:

(a) the real part of $(2i)^{1/2}$

(b) the modulus of the expression $\sum_{n=1}^{N} \mathrm{e}^{in\delta}$ where n is an integer and δ is a real constant

(c) the value of $\cos(2i \ln 2)$. [7]

2. Define the *integration factor* and describe how it is used to solve first order linear differential equations. Using this method or otherwise find the solution $y(x)$ that satisfies the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y^2 - x^2 - 2x}{2y} = 0$$

subject to the boundary condition $y(0) = 1$. [6]

3.

(i) Show that the eigenvalues of a Hermitian matrix are real and that the eigenvectors corresponding to distinct eigenvalues are orthogonal. Explain what it means for an eigenvalue to be degenerate.

(ii) Show that an eigenvalue λ of a unitary matrix must satisfy the condition $|\lambda| = 1$. [7]

4. Consider the set of linear equations in $\mathbb{R}$ with variables x, y, z,

$$\begin{aligned} 3x + y + \lambda z &= 1 \\ x + y + 2\lambda z &= 5 \\ x - z &= -2\mu \end{aligned}$$

where λ and μ are real constants.

(a) Let $\lambda = 2$. For which value of μ will the above equations have a solution? Is this solution unique?

(b) Show that for $\lambda = 1$ and $\mu = \frac{1}{2}$, the set of equations will have a unique solution and find the values of x, y, z corresponding to this solution. [6]

5. Consider the vector $\boldsymbol{\sigma}$ with components $(\sigma_x, \sigma_y, \sigma_z)$ given by the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\hat{\mathbf{n}}$ be a unit vector in $\mathbb{R}^3$ with Cartesian components (n_x, n_y, n_z). Express the components of $\hat{\mathbf{n}}$ in the spherical coordinates (r, θ, ϕ) and hence find the matrix H given by the "dot" product,

$$H = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}} = n_x\sigma_x + n_y\sigma_y + n_z\sigma_z \, .$$

Show that H is Hermitian and find the normalised eigenvectors of the matrix H. [7]

6. The force acting on a charged particle moving at velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ may be written as $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ where q is the particle charge. Write the components of the cross product using the Levi-Civita symbol ϵ_{ijk} defined as

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) \text{ is an even permutation of (1,2,3)} \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of (1,2,3)} \\ 0 & \text{otherwise} \end{cases}$$

and hence show that the components of $\mathbf{F}$ can be expressed in the matrix equation

$$\mathbf{F} = M\mathbf{v}$$

where M is a traceless and antisymmetric matrix. Calculate the elements of the matrix M and show that it is not invertible. [7]

Section B

7.

(a) A form of the Euler differential equation may be written as

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = x \ .$$

Make a suitable substitution to transform the equation into the linear differential equation,

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t} - 2y = \mathrm{e}^t \ .$$

[4]

Find a particular solution for this equation and hence find the general solution $y(x)$ to the Euler equation as given above, subject to the boundary conditions

$$y = 0 \ \text{ for } \ x = 1 \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \ \text{ for } \ x = 1 \ .$$

[7]

(b) Consider the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2y = 2\cos x + 4x\sin x \ .$$

Find a complex function $f(x)$ so that the right hand side of the above equation is the *imaginary* part of $f(x)$. [4]

Using this result or otherwise, find a particular solution $y(x)$ for the differential equation. [5]

8.

(a) Consider the unitary matrix U whose columns are the eigenvectors of the matrix M. Show that U satisfies the relation

$$U^{\dagger} M U = D \ ,$$

where D is a diagonal matrix. [4]

(b) A system of particles is located in the (x, y) plane of a Cartesian coordinate system. The particles are allowed to rotate about an axis in the (x, y) plane, with an angular velocity vector $\boldsymbol{\omega} = (\omega_x, \omega_y)$. The general definition of the kinetic energy of rotation may be written as

$$T(\omega_x, \omega_y) = \frac{1}{2} \boldsymbol{\omega}^T M \boldsymbol{\omega}$$

where M is a 2×2 *symmetric* matrix.

Assume that in a particular case, the kinetic energy of the particles is given by the expression

$$T(\omega_x, \omega_y) = \frac{1}{2} I_0 [2{\omega_x}^2 + 2{\omega_y}^2 - 2\omega_x \omega_y]$$

where I_0 is constant.

(i) Find the elements of the matrix M. [3]

(ii) Find the eigenvalues and normalised eigenvectors of M. Find the elements of the unitary matrix U which is defined by the relation, $U^{-1} M U = D$, where D is a diagonal matrix. [4]

(iii) Show that U is a rotation matrix and find the angle of rotation θ. [3]

(iv) Assume that the angular velocity components are given by the new coordinates $\boldsymbol{\omega'} = (\omega_x', \omega_y')$, where $U\boldsymbol{\omega'} = \boldsymbol{\omega}$. Use the above general definition of the kinetic energy to calculate an expression for the kinetic energy $T'(\omega_x', \omega_y')$.

Show that the constant kinetic energy curve $T'(\omega_x', \omega_y') = 1$ describes an an ellipse in the (ω_x', ω_y') plane and find the major and minor axes of this ellipse. Sketch the constant kinetic energy curve $T(\omega_x, \omega_y) = 1$, showing both the (ω_x, ω_y) and the (ω_x', ω_y') axes, and justify your plots. [6]

9.

(a) Let $\mathbb{L}$: $V \to W$ be a map of the vector space V to the vector space W over a field F. State the conditions for the map $\mathbb{L}$ to be linear. Define the kernel of $\mathbb{L}$ and show that it is a sub–vector space of V. [4]

(b) Using the summation convention or otherwise prove the vector identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$. [5]

(c) Consider the linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined as: $f(\mathbf{r}) = \mathbf{u} \times (\mathbf{r} \times \mathbf{u})$ where $\mathbf{r} \in \mathbb{R}^3$ and $\mathbf{u}$ is a unit vector in $\mathbb{R}^3$. Show that this map is linear and describe the geometry of the image and the kernel. [4]

Let $\mathbf{u} = (u_1, u_2, u_3)$ where the u_i are real numbers. Evaluate the elements of a matrix A that satisfies $f(\mathbf{r}) = A\mathbf{r}$, in terms of u_1, u_2, u_3. Show that the set of three linear equations defined by $A\mathbf{r} = \mathbf{r}$, where $\mathbf{r} = (x, y, z)$, has a non-trivial solution and describe the geometry of the solutions. [7]

10.

(a) Show that any two vectors that are not parallel can form the basis of $\mathbb{R}^2$ and that any three vectors that are not coplanar can form the basis of $\mathbb{R}^3$. [4]

(b) Let the three vectors $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ (which are not necessarily orthogonal nor unit vectors) be a basis set of $\mathbb{R}^3$. The reciprocal vector set $(\mathbf{u}^1, \mathbf{u}^2, \mathbf{u}^3)$ is defined by

$$\mathbf{u}^i \cdot \mathbf{v}_j = \delta_{ij}$$

where δ_{ij} is the Kronecker delta. Show that the reciprocal basis set is also a basis set of $\mathbb{R}^3$ and find expressions for $\mathbf{u}^1, \mathbf{u}^2, \mathbf{u}^3$ in terms of the basis set vectors $\mathbf{v}_i$. [5]

(c) The position vector $\mathbf{r}$ of points on a plane in $\mathbb{R}^3$ may be defined by the relation

$$\mathbf{r} = \mathbf{r_0} + t_1\mathbf{p} + t_2\mathbf{q}$$

where $\mathbf{r_0}, \mathbf{p}, \mathbf{q}$ are constant vectors and t_1, t_2 are scalar parameters.

(i) Explain this definition using sketches showing the vectors $\mathbf{r_0}, \mathbf{p}, \mathbf{q}$. Show that this relation may be expressed in the form $\mathbf{n} \cdot \mathbf{r} = c_0$ where $\mathbf{n}$ is a unit vector perpendicular to the plane and c_0 is a constant.

Write the equation of the plane that passes through the points $(3, 1, -2)$, $(-1, 2, 4)$ and $(2, -1, 1)$, in terms of the coordinates x, y, z. [6]

(ii) The plane described in (i) intercepts a sphere of radius R, centred at the point (x_0, y_0, z_0) resulting in a circle of radius r. Calculate the distance between the centre of the sphere and the centre of the circle and find the radius of the circle. [5]