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FIRST PUBLIC EXAMINATION

Preliminary Examination in Physics

SECOND PUBLIC EXAMINATION

Honour School of Physics,
Parts A and B: 3 and 4 year Courses

SHORT OPTIONS

TRINITY TERM 2016

Tuesday, 14 June
9.30 am to 11.00 am for candidates offering ONE Short Option
9.30 am to 12.30 pm for candidates offering TWO Short Options

Answer **two** *questions from each option for which you have entered.*

Start the answer to each question in a fresh book.

If you have entered for two Short Options,
keep your answers to the two options in different books
and at the end hand in two bundles, one for each option.

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do NOT turn over until told that you may do so.

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Some sections start with a relevant rubric.

Section S1 FUNCTIONS OF A COMPLEX VARIABLE

1. Find the location and residues of the simple poles of the function

$$f(z) = \frac{1}{z^2 \sin \pi(z+\alpha)}$$

where $0 < \alpha < 1$. [8]

Let Γ_N be the square contour with vertices at $N(1+\mathrm{i})$, $N(-1+\mathrm{i})$, $N(-1-\mathrm{i})$, and $N(1-\mathrm{i})$ where N is a positive integer. Explain why

$$\oint_{\Gamma_N} f(z)\,\mathrm{d}z \to 0$$

as $N \to \infty$ and hence show that

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n-\alpha)^2} = \pi^2 \mathrm{cosec}\,\pi\alpha \cot \pi\alpha.$$

[8]

By expanding about $\alpha = 0$ show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

and find

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}.$$

[9]

[You may assume that $\pi^2 \mathrm{cosec}\,\pi\alpha \cot \pi\alpha = \alpha^{-2} - \frac{\pi^2}{6} - \frac{7\pi^4\alpha^2}{120} + O(\alpha^4)$]

2. What is meant by a conformal map? Does a conformal map conserve shapes? Find the condition that the map $z \mapsto w = f(z)$ is conformal. [6]

Let

$$f(z) = \frac{z-1}{z+1}\,.$$

To what region of the w plane is the real axis of the z plane mapped? [4]

Show that a circle of radius r and centre $z = -1$ maps to a circle in the w plane and find its centre and radius. Describe what happens to the *interior* of the circle under the map. [7]

Show that the unit circle in the z plane is mapped to the imaginary axis in the w plane. Which region of the z plane maps to the unit circle in the w plane? [8]

3. (a) Using the method of contour integration evaluate the integral

$$\int_0^{2\pi} \frac{1}{3 + 2\cos\theta + \sin\theta}\,\mathrm{d}\theta\,.$$

[10]

(b) State Cauchy's Theorem and derive the criterion satisfied by the contours Γ and Γ' for the function $f(z)$ if

$$\oint_\Gamma f(z)\,\mathrm{d}z = \oint_{\Gamma'} f(z)\,\mathrm{d}z\,.$$

[6]

Give the location and type of the singular points of the function $f(z) = (z^2 - 1)^{1/2}$. The contour Γ encircles the segment $[-1, 1]$ of the real axis. By considering

$$\oint_\Gamma f(z)\,\mathrm{d}z$$

evaluate

$$\int_{-1}^{1} \sqrt{1 - x^2}\,\mathrm{d}x.$$

[9]

Section S2 ASTROPHYSICS: FROM PLANETS TO THE COSMOS

1. State Kepler's first two laws of planetary motion. By considering two stars of mass M_1 and M_2 moving on circular orbits around a common centre of mass with period P, derive an expression for Kepler's third law,

$$\frac{a^3}{P^2} = \frac{G}{4\pi^2}(M_1 + M_2),$$

where a is the separation of the two stars. [8]

An eclipsing spectroscopic binary star has a period of 5.45 days. Radial velocity measurements show the two stars moving with equal and opposite amplitudes of $100\,\mathrm{km\,s^{-1}}$. What is the separation of the stars in Astronomical Units? Calculate the masses of the stars in solar units assuming they are moving on circular orbits. [6]

When the primary star is eclipsed by the secondary, the light from the binary is observed to decrease, reaching a minimum value after 1.7 hours. The duration of minimum light is 0.8 hours, after which the binary returns to its original brightness 1.7 hours later. Draw a sketch of this light curve, showing the relative positions of the stars at first contact, the beginning and end of totality, and last contact. Calculate the radii of the two stars in kilometres. The smaller star is on the Main Sequence; where on the Hertzsprung-Russell diagram is the larger star? [11]

2. Write down the equation of hydrostatic equilibrium for a star. By considering the free-fall time from the edge to the centre of the Sun, estimate the dynamical timescale of the Sun.

Write down the Virial theorem connecting the total thermal energy of a star, U, to its gravitational potential energy, Ω. Derive an expression for the minimum total gravitational potential energy of the Sun. Use the Virial theorem to estimate the lifetime of the Sun if gravitational contraction is the only source of energy. [13]

The reaction which converts hydrogen to helium in the core of the Sun can be summarised as:

$$4p \rightarrow He + 2e^+ + 2\nu_e + 2\gamma.$$

Calculate the amount of energy released in this reaction as a fraction of the total input energy [$\mathrm{m}_H = 1.0078$ amu. and $\mathrm{m}_{He} = 4.0026$ amu.].

Assuming the present constant luminosity, and that the Sun moves away from the Main Sequence when it has converted 10% of its mass to helium, estimate the Main Sequence lifetime of the Sun.

Calculate the ratio of the potential energy of Coulomb repulsion between two protons separated by $10^{-15}\,\mathrm{m}$, and the average kinetic energy of the protons in the core of the Sun where the temperature is $2 \times 10^7\mathrm{K}$. How is it possible for the above reaction to fuel energy generation in the Sun? [12]

3. Draw diagrams of the microwave sky in galactic co-ordinates to illustrate :

(a) the emission from dust in the Milky Way,

(b) the dipole motion of the Milky Way,

(c) the structure of the cosmic microwave background.

Briefly describe each diagram. What is the temperature of the microwave background seen by an observer in a galaxy at a redshift, $z = 2$?

One side of the microwave sky is 0.105% hotter and the opposite side 0.105% cooler than the average temperature. Calculate the speed of the Milky Way with respect to the background. Is the Milky Way moving towards or away from the hotter region? [14]

Cepheid variable stars obey the period-luminosity relation, $M_V = -3.5 \log_{10} P$, where M_V is the V-band mean absolute magnitude of the star and P is its period in days. Cepheid variables in the Virgo cluster are measured to have a mean apparent magnitude of $V = 25.5$ and a period of 50 days. What is the distance to the Virgo cluster in megaparsecs (Mpcs)? Given the Hubble constant is $H_0 = 72$ km s^{-1} Mpc^{-1}, what recession velocity would you expect for the Virgo cluster based on the distance you have calculated? How do you interpret the difference between this value and the measured recession velocity for the Virgo cluster of 1150 km s^{-1}? [11]

Section S3 QUANTUM IDEAS

1. At the end of the 19th century there were some puzzling phenomena that could not be described using a continuous wave theory of light. Name **two** and explain how the issues associated with these were resolved using the concept of photons. [5]

A beam of helium atoms with transverse width $a = 1$ μm travels in the z direction with kinetic energy $T = 5\times10^{-3}$ eV. What is the de Broglie wavelength, λ, of the helium atoms? Estimate the uncertainties in the transverse momentum, Δp_x, and position, Δx, of the beam. [Assume helium atoms comprise 2 protons and 2 neutrons.] [3]

The helium beam is incident on a slit of width $w = 64$ nm in the x direction centred at the origin. Calculate the transverse momentum wave function, $\tilde{\psi}(p_x)$, after the slit. Thus show that the normalized transverse momentum probability distribution is

$$\tilde{p}(p_x) = \frac{1}{\delta p_x}\mathrm{sinc}^2\left(\pi\frac{p_x}{\delta p_x}\right),$$

where $\mathrm{sinc}(\alpha) = \sin(\alpha)/\alpha$, $p_x = p\sin\theta$, θ is the angle made by the total momentum with respect to the z axis, and p is the magnitude of total momentum of the atom. Provide an expression for δp_x. [6]

Determine the probability for a helium atom to pass through a slit of width $d \ll w$ placed just after the first slit of width w. Give an expression for the transverse momentum probability distribution, $\tilde{p}(p_x)$, and corresponding uncertainty, Δp_x, after passing through the second slit of width d. Show that this result is consistent with the Heisenberg uncertainty principle. How would your results differ if the experiment were performed with X-rays with energy 10 keV? [7]

To encode $b = 6$ bits of information onto the transverse position of the helium atom beam after the slit of width w, what width d should detector pixels be? Is it feasible to similarly encode 10 bits of information in the helium atom? [4]

$$\left[\text{You may use the following integral: } \int_{-\infty}^{+\infty}\mathrm{sinc}^2(\alpha)\,\mathrm{d}\alpha = \pi.\right]$$

2. Write down the time-dependent Schrödinger equation for a particle of mass m free to move only along the x axis in a region with potential energy $V(x)$. Give a physical interpretation of each term and show how the time-independent Schrödinger equation (TISE) is obtained for a particle of well-defined energy. Give a physical interpretation of the particle wave function $\psi(x,t)$. What are the dimensions of the wave function $\psi(x,t)$? [7]

Consider a particle constrained to move along the x direction subject to a restoring force $F = -kx$.

(a) Write down the TISE and sketch the corresponding potential for this case. [3]

(b) The ground and first excited state wave functions for this system are

$$\phi_0(x) = \frac{1}{\sqrt{\pi^{1/2}\sigma}} \exp\left(-\frac{x^2}{2\sigma^2}\right) \qquad \text{and} \qquad \phi_1(x) = \sqrt{\frac{2}{\pi^{1/2}\sigma^3}}\, x \exp\left(-\frac{x^2}{2\sigma^2}\right),$$

where $\sigma = \sqrt{\hbar/m\omega}$ and $\omega = \sqrt{k/m}$. Using the TISE calculate the energies, E_0 and E_1, associated with these states expressed in terms of $\hbar\omega$. [4]

(c) Give expressions for the probability distributions, $p_0(x)$ and $p_1(x)$, for these states and plot them on separate graphs. What is the probability to find a particle in a small region δx centred on the origin when the particle is prepared in each state? [4]

(d) Calculate the average position $\langle x \rangle$ and corresponding uncertainty in position $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ for both states. Indicate these on your plot of the probability distributions. [3]

(e) For a particle prepared in a superposition of the ground and first-excited states, $\phi_+(x) = [\phi_0(x) + \phi_1(x)]/\sqrt{2}$, at time $t = 0$, determine the minimum time required to evolve into the orthogonal state $\phi_-(x) = [\phi_0(x) - \phi_1(x)]/\sqrt{2}$. [4]

$$\left[\text{You may assume: } \int_{-\infty}^{+\infty} x^2 \exp\left(-\frac{x^2}{\sigma^2}\right) \mathrm{d}x = \frac{\sqrt{\pi}\sigma^3}{2} \text{ and } \int_{-\infty}^{+\infty} x^4 \exp\left(-\frac{x^2}{\sigma^2}\right) \mathrm{d}x = \frac{3\sqrt{\pi}\sigma^5}{4}.\right]$$

3. Describe the Franck-Hertz experiment and explain how it provides evidence for discrete atomic energy levels. Provide a sketch of the experiment and a rough plot of typical observed data in your answer. [5]

Discuss the stability of the orbit of an electron orbiting a nucleus when treated using classical physics. State the assumptions made by Bohr in his model of ionized hydrogen-like atoms with a single electron of mass m_e and charge $-e$ orbiting a nucleus with charge Ze and mass M, that enabled him to address this problem. What constraint does this model impose on the de Broglie wavelength of the electron? [5]

Derive expressions for the allowed radii and energy levels in terms of the Bohr radius, a_0, for such a hydrogen-like ion using Bohr's assumptions. What are the corresponding expressions for the allowed radii and energy levels of positronium, consisting of an electron e^- and positron e^+? [6]

Show that the period of revolution for an electron in the n^{th} excited state of the hydrogen-like ion in Bohr's model is $T_n = 4\epsilon_0^2 h^3 n^3 / Z^2 m e^4$. What is the corresponding expression for positronium? [4]

Calculate the wavelength in nm of the Lyman-α line (the transition from the first excited state to the ground state) for (a) a hydrogen-like lithium ion ($Z = 3$) and (b) positronium and explain the origin of their difference. [5]

Section S4 ENERGY STUDIES

1. Describe the physical principles of a deuterium-tritium tokamak fusion reactor and its main components. [10]

Explain the role of lithium in D-T fusion reactors. [3]

If extracted at large scale, it is estimated that Li could be extracted from seawater, where Li is at 0.17ppm, at about 2.5kWh$_e$ per gram of Li. Estimate whether this makes energetic sense if a future 2GW$_e$ fusion power plant would use about 300kg D and about 6t of Li per year. For approximately how long could Li extracted from seawater power the world? [5]

For a fusion reactor using a 50-50 mixture of D and T, the Lawson Criterion turns out to be $n\tau_{\mathrm{E}}T \geq 3 \times 10^{21}\mathrm{m}^{-3}\,\mathrm{s\,keV}$ (T in keV). In such a reactor with $\tau_{\mathrm{E}} = 2.5$s, the plasma pressure is $1.35 \times 10^5\mathrm{N/m}^2$ and the average kinetic energy of the particles is 12.9keV. Compare this to the Lawson Criterion and comment on the mode of operation of this fusion reactor. [7]

2. Derive how the flow over a broad-crested weir can be used to determine the flow rate per unit width, Q, when the depth of water upstream is much larger than the depth of water over the weir. [7]

Under water at the base of the weir is a horizontal slit of height d ($d \ll$ height of the weir) that can be opened, such that no more water is flowing over the weir. Instead, the water flowing through the slit is diverted to Pelton turbines. Describe how a Pelton turbine works and derive that its maximum power per unit width of weir, $P_{\max}$, is given by

$$P_{\max} = \frac{1}{2}\rho Q u^2 = \frac{1}{2}\frac{\rho}{d^2} g^{3/2} \left(\frac{2}{3} y_{\min}\right)^{9/2},$$

where ρ is the density of water, u the speed with which the water is leaving the slit and striking the turbine, g the acceleration due to gravity and y_{min} the height of the undisturbed level of water above the top of the weir and far upstream when $d = 0$. [10]

Outline the main advantages and disadvantages of hydro-power. [8]

3. Comment on and discuss *critically*, with numerical estimates where appropriate, **three** of the following statements:

(a) Solar Photovoltaics will never be able to compete with coal for electricity generation.

(b) The highest priority for CO_2 reductions is decarbonisation of the electricity sector.

(c) If every household in the UK would harvest the power from rain running down from their roofs using micro water turbines, we would save a significant amount of CO_2.

(d) Renewable energies will never be able to provide base load energy.

(e) Increasing the energy efficiency of all fossil power plants in China to state-of-the-art will save more CO_2 than making the UK 100% renewable.

(f) Lithium ion batteries are the best energy storage technology.

(g) What are the economical, environmental and waste issues challenging nuclear fission?

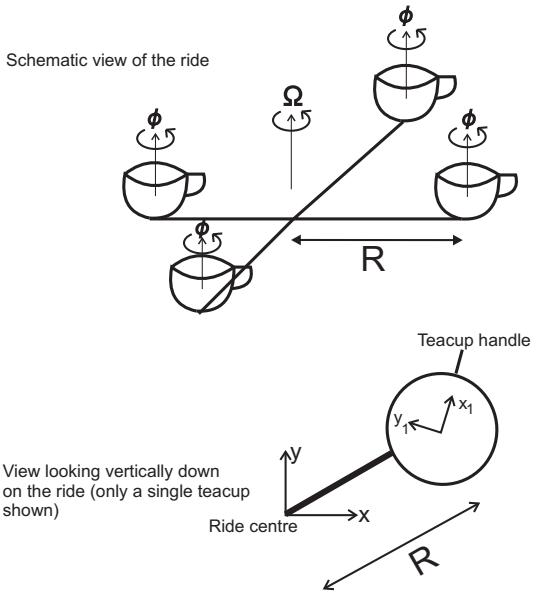
[25]

Section S7 CLASSICAL MECHANICS

1. State the form of the Coriolis force. Using this, determine the direction of rotation of air for both low-pressure and high-pressure storm systems in Earth's northern hemisphere. Determine whether the Great Red Spot, which rotates anti-clockwise in the southern hemisphere of Jupiter, is a low or high-pressure system. [5]

A fairground ride consists of giant teacups, whose centres are at a distance R from the overall centre of the ride. The ride rotates at an angular velocity of Ω, with the teacups also simultaneously rotating about their own central axis with an angular velocity of ϕ, such that the time interval between the teacup handle pointing in the radial direction is $2\pi/\phi$.

A diagram of the ride, viewed from the vertical direction, is shown below. The coordinates (x, y) are inertial coordinates with origin at the centre of the ride. The coordinates (x_1, y_1) have their origin at the centre of a teacup, and are aligned such that coordinate x_1 always points along the handle of the teacup. At $t = 0$, (x_1, y_1) have the same orientation as (x, y). The internal surface of the teacup satisfies the equation $z = \omega^2/(2g)(x_1^2 + y_1^2)$.



Relate the coordinates (x_1, y_1) to the coordinates (x, y) and show that, when $\phi = -\Omega$, these relations reduce to

$$x = R\cos\Omega t + x_1, \qquad y = R\sin\Omega t + y_1.$$

Interpret these equations. [As an intermediate step, you may find it useful to introduce coordinates (x', y') that are centred on the teacup, with x' always pointing radially outwards from the centre of the ride.] [8]

A small child riding inside the teacup drops a marble of mass m on the floor. Using (x_1, y_1) to label the position of the marble, and neglecting any vertical velocities, determine the Lagrangian for the marble. [4]

Calculate the generalised momenta, and solve for the motion of the marble, $(x_1(t), y_1(t))$. [8]

2. What is a constant of motion? State and prove Noether's theorem. What is the significance of symmetries of the Lagrangian? [8]

Consider a system of three particles, of masses m_1, m_2 and m_3, moving and interacting in three-dimensional space (x, y, z). The position of each particle is labelled by $\mathbf{r}_i \equiv (x_i, y_i, z_i)$, for $i = 1, 2, 3$. The Lagrangian describing the system is

$$L = \sum_{i=1}^{3} \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 - \sum_{i=1}^{3} \frac{1}{2} m_i \omega^2 x_i^2 - \cos(\lambda(m_1 - m_2)t) U_{12}(y_1 - y_2) - U_{13}(y_1 - y_3) - V(z_1 - z_2)$$

You may regard U_{12}, U_{13} and V as generic functions with no special properties. ω and λ are constants that are independent of the masses m_i.

(a) In the case that $m_1 \neq m_2 \neq m_3$, identify and describe the conserved quantities. [5]

(b) How many additional conserved quantities exist in the cases that, first, $m_1 = m_2$ and secondly, $m_1 = m_2 = m_3$? [7]

(c) In the latter case where all masses are equal, write explicit expressions for these additional conserved quantities and interpret them. [5]

3. Starting with the action principle, derive the Euler-Lagrange equations,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0.$$

Define the canonical momentum, p_i, and the Hamiltonian, $H(\mathbf{q}, \mathbf{p}, t)$. State Hamilton's equations and derive them using the action principle. [11]

Explain what is meant by *phase space*, and state and prove Liouville's theorem. [6]

Illustrate Liouville's theorem by explicitly verifying it for motions in the phase space of the harmonic oscillator Hamiltonian,

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}.$$ [4]

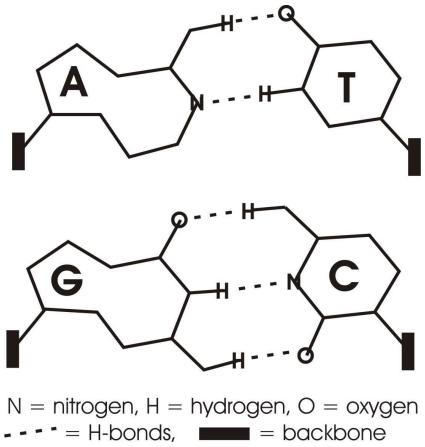
A high-energy particle collider operates by colliding counter-rotating bunches of protons. Each bunch may be considered isolated and consists of N protons. Collisions occur as counter-rotating bunches pass through each other. To maximise the rate of collisions, it is desirable to 'cool' the bunches by minimising the overall spread among the particles in each bunch of positions and momenta transverse to the beam axis.

Analyse this problem qualitatively from the perspective of Liouville's theorem. Your answer should include a description of the apparent problems in performing this cooling and suggestions for ways that it can nonetheless be done. [4]

Section S12 INTRODUCTION TO BIOLOGICAL PHYSICS

1. Describe how the structures of nucleic acids and proteins relate to their functions. Your answer should include a statement and discussion of the "Central Dogma" of molecular biology. [25]

2. Describe the essential elements of Darwin's theory of Natural Selection and the evolution of species. [5]



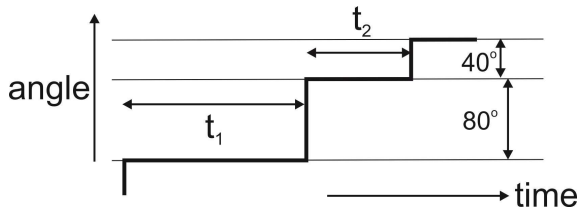
Base-pairs containing one purine (A or G) and one pyrimidine (C or T) fit into the DNA double helix as shown above. Considering only the contribution of hydrogen bonding to base-pair stability, list all possible mis-matched purine-pyrimidine base-pairs and estimate the probabilities that each will be formed relative to correct pairs. [You may assume that the relative positions of the 5- and 6-membered rings of mismatched pairs are approximately as shown.] Thus, make a rough prediction of the number of base-pair errors expected in copying the entire human genome. Evolutionary mutations appear at a rate of approximately one base pair per human generation. Compare your answer to this rate and explain any differences. [A typical hydrogen bond energy is ~ 5 times the thermal energy, k_BT.] [8]

An m-state information storage element can store n "bits", where $log_2(m) = n$. Make a simple sketch of B-DNA, labelled with approximate dimensions, and use this to estimate an upper limit to the information storage capacity of B-DNA, giving your answer in units of bits m^{-3}. Compare your answer to the maximum information density stored in DNA in living cells and discuss the discrepancy. [8]

There are 3 stop codons in the genetic code. What is the probability that a codon in a random DNA sequence will be a stop codon? Hence, derive an expression for the probability that a random DNA sequence will encode a protein of length L or more amino acids. What is the probability of such a protein exceeding 300 amino acids in length? [4]

3. Make a labelled sketch of the enzyme ATP-synthase, and describe briefly its function and mechanism. [10]

The α- and β-subunits of single molecules of F_1-ATPase can be immobilized in a light microscope, and rotation of the γ-subunit driven by ATP hydrolysis measured with high spatial and temporal resolution. Representative data obtained under these conditions, at intermediate ATP concentrations, are sketched below.



The distributions of time intervals shown fit well to the following functions:

$$\begin{aligned} P_1(t_1) &= k_1 c \mathrm{e}^{-ck_1t_1} \\ P_2(t_2) &= \left(\frac{k_a k_b}{k_a - k_b}\right)\left(\mathrm{e}^{-k_b t_2} - \mathrm{e}^{-k_a t_2}\right), \end{aligned}$$

where c is the ATP concentration and k_1, k_a and k_b are constants.

What can be inferred from this observation about the mechanism of ATP hydrolysis by F1-ATPase? Your answer should include derivations and sketches of the distributions P_1 and P_2. [15]

Section S16 PLASMA PHYSICS

1. Stating any assumptions that you make, show that when atomic hydrogen is heated in thermodynamic equilibrium, the free-electron number density, n_e, and the number density of neutral hydrogen atoms, n_{H}, satisfy the Saha equation:

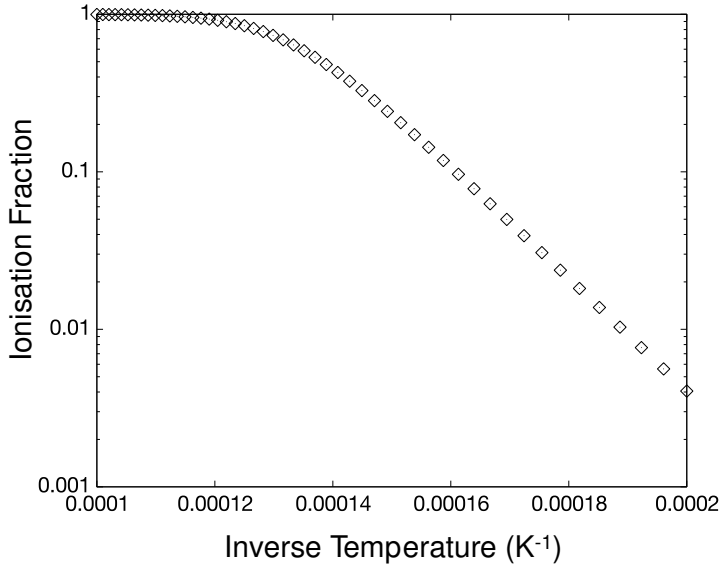
$$\frac{n_{\mathrm{e}}^2}{n_{\mathrm{H}}} = \frac{(2\pi m_{\mathrm{e}} k_{\mathrm{B}} T)^{3/2}}{h^3} \exp\left(-R/k_{\mathrm{B}}T\right),$$

where R is the ionisation potential of atomic hydrogen (the Rydberg) and ideal gas partition functions have been used. [10]

The ionisation fraction, ζ, is defined as the ratio of the number of free electrons to the total number of protons (both free protons and those bound to an electron). Show that, for low degrees of ionisation, where $\zeta \ll 1$,

$$\zeta \approx AT^m \exp(-T_{\mathrm{H}}/T),$$

where A is a constant that need not be determined. Deduce values for m and T_{H}. [6]



The figure above shows a plot of the ionisation fraction, ζ, as a function of inverse temperature for a hydrogen plasma. Verify that the value of T_{H} you have deduced is consistent with this plot, and determine the number density of hydrogen atoms at low temperature. [9]

2. Show that, in a plasma with electron number density, n_e, and temperature, T, the charges of ions are shielded on the scale of the Debye length,

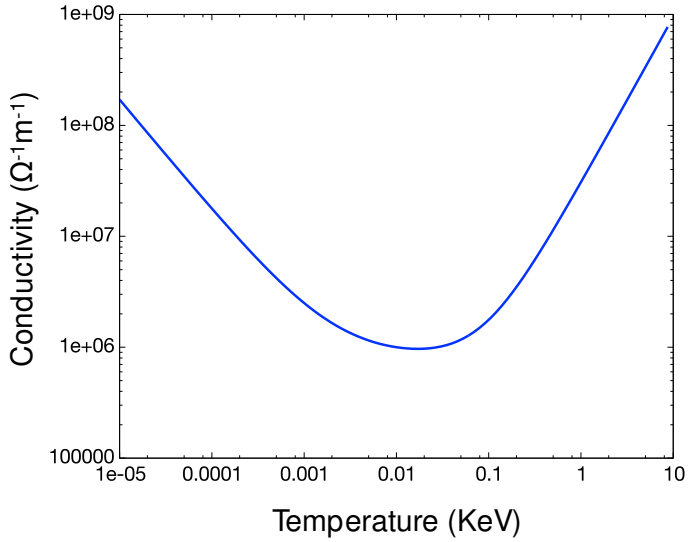
$$\lambda_\mathrm{D} = \sqrt{\frac{\varepsilon_0 k_\mathrm{B} T}{n_\mathrm{e} e^2}},$$

and that the natural frequency of oscillation of the electrons is given by the plasma frequency, ω_p,

$$\omega_\mathrm{p} = \sqrt{\frac{n_\mathrm{e} e^2}{\varepsilon_0 m}}.$$ [6]

State what is meant by the *plasma parameter*, N_D. Assuming that a reasonable approximation of the collision time, τ, of an electron in a plasma with ions of charge Z is $\tau \approx N_\mathrm{D}/(\omega_\mathrm{p} Z)$, obtain an expression for τ. The electrical conductivity of a system, σ, due to electrons of number density, n_e, which have an effective collision time τ, is given by $\sigma = n_\mathrm{e} e^2 \tau / m$. Show that the electrical conductivity of a plasma increases with temperature and, within this simplified approximation, is independent of the electron density. [9]

In contrast to a hot plasma, for a cold metal the scattering of electrons from quantised vibrations in the lattice (phonons) results in a collision time that is inversely proportional to temperature. The figure below shows the electrical conductivity of solid density aluminium ($Z = 13$, $\rho = 2.7$ g cm^{-3}) as a function of temperature (given in keV), as it passes from a cold solid to a hot plasma. Explain the qualitative form of the plot. [4]



Deduce the value of the temperature, T', at which the electrical conductivity of the hot plasma is equal to that of the cold solid at room temperature. Make an estimate of the plasma parameter at T' and comment on your result. [6]

3. The Vlasov equation describes the distribution functions, f, of electrons and ions in a plasma. For an unmagnetised plasma it is given by

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{q}{m}(\mathbf{E}) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0,$$

and the equation holds for the electrons and the ions separately. Consider a plasma at a temperature T, with electrons and ions with a corresponding Maxwellian distribution, f_0, where

$$f_0 = \frac{n_0}{\pi^{3/2}} \left(\frac{m}{2k_\mathrm{B}T} \right)^{3/2} \exp\{-m(v_x^2 + v_y^2 + v_z^2)/(2k_\mathrm{B}T).$$

Electron waves of angular frequency, ω, and wavevector, $\mathbf{k}$, propagate in the plasma, leading to a perturbation of the electron distribution function:

$$f_1 = f_{1(\mathbf{k},\omega)} \exp(i[\mathbf{k} \cdot \mathbf{r} - \omega t]).$$

Assuming that the waves propagate along the z direction, show that they obey the dispersion relation

$$1 - \frac{2n_0 e^2}{\pi^{1/2}\varepsilon_0 mk} \left(\frac{m}{2k_\mathrm{B}T} \right)^{3/2} \int \frac{v_z \mathrm{e}^{-\frac{mv_z^2}{2k_\mathrm{B}T}}}{\omega - kv_z} \mathrm{d}v_z = 0.$$

Briefly describe the physical significance of the pole in the integral. [12]

Consider plasma waves with a phase velocity far greater than the velocity of a typical thermal electron ($v_\mathrm{th} = \sqrt{(k_\mathrm{B}T/m)}$). Show that these waves obey the Bohm-Gross formula

$$\omega^2 = \frac{ne^2}{\varepsilon_0 m} + 3k^2 v_\mathrm{th}^2.$$ [8]

Show that the product of the group velocity, v_g, and phase velocity, v_ϕ, is given by $v_g v_\phi = 3v_\mathrm{th}^2$, and hence explain why it is uncommon to find waves in plasmas with group velocities significantly greater than the thermal velocity. [5]

Section S18 ADVANCED QUANTUM MECHANICS

1. A non-relativistic quantum particle of mass m is moving in the one-dimensional potential $U(x) = -\alpha\,[\delta(x-a)+\delta(x)+\delta(x+a)]$, where $\alpha > 0$, $a > 0$. Using the Green's function, $G(x,x') = m/(\kappa\hbar^2)\exp(-\kappa|x-x'|)$, of the Schrödinger operator for a free particle with $E<0$, where $\kappa = \sqrt{-2mE}/\hbar$, show that the Schrödinger equation,

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + U(x)\psi(x) = E\psi(x),$$

for the particle in the potential $U(x)$ with $E<0$, subject to the wave function boundary conditions $\psi(x)\to 0$ for $x\to\pm\infty$ corresponding to bound states in the potential $U(x)$, can be written as an integral equation

$$\psi(x) = -\frac{m}{\kappa\hbar^2}\int\limits_{-\infty}^{\infty} \mathrm{e}^{-\kappa|x-x'|}U(x')\psi(x')dx'\,.$$

Using this integral equation and the explicit form of the potential, write down the solution $\psi(x)$ and the set of three algebraic equations determining $\psi(a)$, $\psi(0)$ and $\psi(-a)$. Express these equations in matrix form. Show that the bound state energies are determined by the equation

$$1+\eta\lambda - 2\mathrm{e}^{2\eta}(1-\eta\lambda) + \mathrm{e}^{4\eta}(1-\eta\lambda)^3 = 0\,,$$

where $\eta = \kappa a$, $\lambda = \hbar^2/\alpha m a$. [9]

Analyse the equation for bound states energies in the limit of small and large λ. How many energy levels are there for $\lambda \ll 1$? Find their approximate values in terms of λ and κ. Find the energy levels and their degeneracy for $\lambda \gg 1$. [5]

Now consider stationary states of the continuous spectrum with $E>0$ in the same potential $U(x)$. For a particle incident on the potential from the negative x direction with momentum, k, and obeying the integral equation

$$\psi_s(x) = \mathrm{e}^{\mathrm{i}kx} - \frac{\mathrm{i}m}{k\hbar^2}\int\limits_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}k|x-x'|}U(x')\psi_s(x')dx'\,,$$

find the solution $\psi_s(x)$ corresponding to stationary scattering states. By analyzing the asymptotics of the solution for $x\to+\infty$ and writing it in the form $S(k)\mathrm{e}^{\mathrm{i}kx}$, find the scattering amplitude $S(k)$ in terms of $\psi_s(a)$, $\psi_s(0)$ and $\psi_s(-a)$. [4]

Find the system of three algebraic equations determining $\psi_s(a)$, $\psi_s(0)$ and $\psi_s(-a)$ and write them in matrix form. Use variables λ and $\xi = ka$. Show that the singularities of $S(k)$ on the imaginary axis of ξ (i.e. for $\xi = i\eta$) are determined by an algebraic equation identical to the one determining the bound states energy levels in the potential $U(x)$. Do you expect $S(k)$ to have other singularities? If yes, explain their physical significance. [7]

2. A spinless free relativistic particle of mass m obeys the Klein - Gordon equation

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta + \frac{m^2c^2}{\hbar^2}\right]\psi = 0\,,$$

where Δ is the Laplacian in three space dimensions. What is the connection between this equation and the dispersion relation $E^2 = p^2c^2 + m^2c^4$ for the relativistic particle? [2]

A spinless relativistic particle of mass m in an external scalar potential $U(\mathbf{r}, t)$ obeys the equation

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta + \frac{m^2c^2}{\hbar^2} + \alpha\, U(\mathbf{r}, t)\right]\psi = 0\,.$$

By considering the non-relativistic limit of the associated dispersion relation, show that $\alpha = 2m/\hbar^2$. [2]

A spinless relativistic particle of mass m and charge e in an external electromagnetic field $A^\mu = (\Phi, \mathbf{A})$ obeys the equation

$$\left[c^2\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)^2 - \left(i\hbar\frac{\partial}{\partial t} - e\Phi\right)^2 + m^2c^4\right]\psi = 0\,,$$

where $\hat{\mathbf{p}} = -\mathrm{i}\hbar\nabla$. Show that the current density

$$j_\mu = -\frac{\mathrm{i}}{2}\left(\psi\partial_\mu\psi^* - \psi^*\partial_\mu\psi\right) - \frac{e}{\hbar c}A_\mu\psi^*\psi\,,$$

where ψ is a solution of the Klein-Gordon equation, satisfies the continuity equation $\partial_\mu j^\mu = 0$.

For a time-independent electromagnetic field, consider solutions of the form $\psi(t, \mathbf{r}) = \mathrm{e}^{-\mathrm{i}(mc^2+E)t/\hbar}\varphi(\mathbf{r})$. Show that the stationary Klein-Gordon equation obeyed by $\varphi(\mathbf{r})$ is

$$\left[c^2\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)^2 + m^2c^4\right]\varphi = \left(mc^2 + E - e\Phi\right)^2\varphi\,.$$

[8]

For a hydrogen-like atom $e\Phi = -Ze^2/r$, $\mathbf{A} = 0$. Show that the radial part $R(r)$ of the wave function $\varphi = R(r)Y_{lm}$, where $Y_{lm}(\theta, \phi)$ are the standard spherical harmonics, obeys the equation

$$\frac{1}{r}\frac{d^2(rR)}{dr^2} - V_{eff}R(r) = A\,R(r)\,,$$

where $V_{eff} = (l(l+1) - \alpha_{em}^2Z^2)/r^2 - 2B/r$ and $\alpha_{em} = e^2/\hbar c$. [*Hint: You may use the decomposition* $\Delta\varphi = 1/r\,\mathrm{d}^2(rR)/\mathrm{d}r^2Y_{lm} - [l(l+1)]/r^2RY_{lm}$] Find the constants A, B and discuss their non-relativistic limit. [8]

Sketch the effective potential V_{eff}. Explain qualitatively whether you expect bound states to exist for any value of Z. Do you expect the energy spectrum derived from this equation to be in agreement with experiment? Explain. [5]

3. Consider the following candidate for a wave equation describing a relativistic particle of mass m

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar c}{\mathrm{i}}\left(\alpha_1\frac{\partial\psi}{\partial x^1} + \alpha_2\frac{\partial\psi}{\partial x^2} + \alpha_3\frac{\partial\psi}{\partial x^3}\right) + \beta mc^2\psi \equiv H_D\psi\,.$$

Can the coefficients α_i, $i = 1, 2, 3$ and β be real numbers? Explain. Assuming that ψ has N components ψ_α, $\alpha = 1, ...N$, show that each component satisfies the Klein-Gordon equation,

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta + \frac{m^2c^2}{\hbar^2}\right]\psi_\alpha = 0\,,$$

provided that $\alpha_i^2 = I$, $\beta^2 = I$, $\alpha_i\alpha_k + \alpha_k\alpha_i = 2\delta_{ik}$ and $\alpha_i\beta + \beta\alpha_i = 0$, where I is the identity matrix.

Show that the eigenvalues of α_i and β are equal to ± 1. Show that $\mathrm{Tr}\,\alpha_i = 0$ and $\mathrm{Tr}\,\beta = 0$. Show that N must be an even number greater than 2. [10]

For $N = 4$, introducing $\psi^\dagger = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*,)$, where ψ_α^* is the component complex conjugate to ψ_α, using the wave equation for ψ_α and its Hermitian conjugate, show that the continuity equation holds,

$$\frac{\partial\rho}{\partial t} + \nabla.\mathbf{j} = 0\,,$$

where $\rho = \psi^\dagger\psi$, $j_k = c\psi^\dagger\alpha_k\psi$. [5]

The Dirac equation in an external electromagnetic field $A^\mu = (\Phi, \mathbf{A})$ is

$$\left[\gamma^\mu\left(p_\mu - \frac{e}{c}A_\mu\right) - mc\right]\psi = 0\,,$$

where $\psi = \begin{pmatrix}\varphi\\ \chi\end{pmatrix}$ is the four-component Dirac spinor. The Minkowski metric is given by $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$, $p_\mu = \mathrm{i}\hbar\partial_\mu$, and the Dirac matrices are

$$\gamma^0 = \begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k\\ -\sigma^k & 0\end{pmatrix},$$

where σ^k are Pauli matrices obeying $\sigma_i\sigma_k = \delta_{ik} + \mathrm{i}\epsilon_{ikl}\sigma_l$. For time-independent external fields, consider stationary solutions of the form $\psi \sim \exp\left[-\mathrm{i}(mc^2 + E)t/\hbar\right]$. Write down the system of coupled equations for the spinors φ and χ. [3]

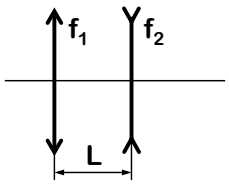
Show that in the non-relativistic limit $|E| \ll mc^2$, $|e\Phi| \ll mc^2$, the spinor φ obeys the Pauli equation

$$\mathrm{i}\hbar\frac{\partial\varphi}{\partial t} = \left[\frac{\left(\mathbf{p} - e/c\mathbf{A}\right)^2}{2m} + e\Phi - \mu_0\sigma\cdot B\right]\varphi$$

and find the value of the magnetic moment μ_0. Does the Dirac equation uniquely predict the value of μ_0? Explain. [7]

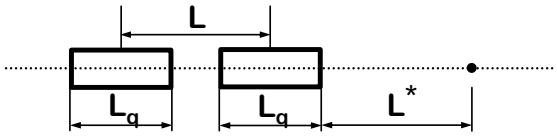
Section S19 UNIFYING PHYSICS OF ACCELERATORS, LASERS AND PLASMA

1. Two thin quadrupoles are separated by a distance L and their focal lengths are f_1 and f_2. They are used as a final doublet for the Future Circular Collider (FCC) to focus the proton beam of energy E=50 TeV. A small additional focusing of the proton beam can also be provided by the electron lens.

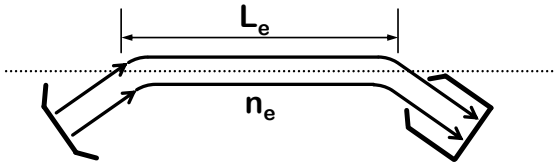


(a) Define the region of range of focal lengths where a pair of such quadrupoles will focus the beam in both planes. Express the results in terms of the inverse normalized focusing distance of the quadrupoles $x_1 = L/f_1$ and $x_2 = L/f_2$. [9]

(b) Based on these results provide an estimate of the parameters of the final doublet for FCC (i.e. length of the quadrupoles L_q and distance between the centers of the quadrupoles L), taking into account that the gradient of the quadrupoles G cannot exceed 300 T/m and that the desired distance from the focal point to the nearest quadrupole in FCC is $L^* = 50$ m. Assume that the absolute strength of the quadrupoles is the same and that you can simplify your expression assuming that the lengths L or L_q are much shorter than L^*. [9]



(c) Estimate parameters of the electron lens (electron density n_e and length of the beam L_e) needed to produce 0.01% correction to the focusing of the final doublet found in the previous task. Assume that the electron beam is not relativistic and that the electron lens is placed in front of the final doublet. [7]



2. Consider a 3 GeV electron storage ring with a horizontal emittance of $\varepsilon_x = 3$ nm. Assume the optics functions at the centre of the undulators are $\beta_x = \beta_y = 2$ m, $D_x = D_y = D_x = D_y = \alpha_x = \alpha_y = 0$ and that the vertical emittance is 1% of the horizontal emittance.

(a) Compute the electron beam size and divergence in horizontal and vertical dimensions [4]

(b) Compute the photon beam size and divergence for an undulator with L = 2 m emitting in the fundamental wavelength at $\lambda = 10$ nm [3]

We want to increase the photon beam brightness by building a new lattice to reduce the horizontal emittance by a factor 10, to 300 pm. We use the same optics functions at the centre of the undulator and the same ratio between horizontal and vertical emittance.

(c) What is the percentage increase in brightness at $\lambda = 10$ nm after reducing the emittance as indicated above? [8]

(d) At what wavelengths does the increase in brightness becomes negligible? [3]

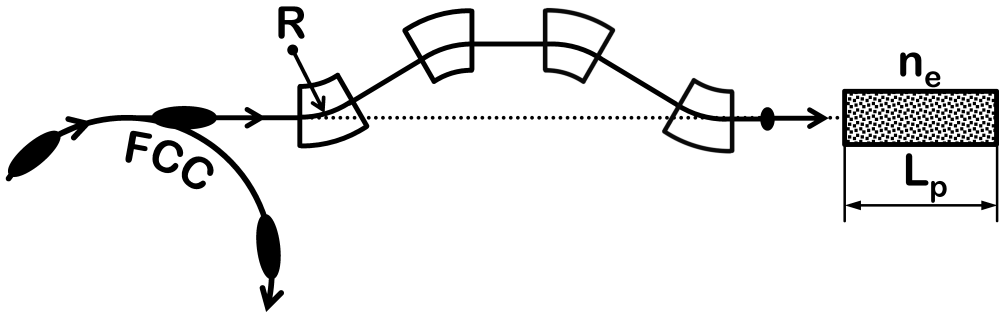
We now want to achieve the emittance reduction only by reducing the energy of the ring.

(e) What should be the new electron beam energy? [3]

(f) Would this be a good strategy to increase the brightness in a light source? Explain why. [4]

[Hint: What happens to the spectrum of the bending radiation? And the undulator radiation?]

3. Assume that the FCC proton beam of $E = 50$ TeV is going to be used as a driver for the plasma acceleration of a short sub-mm electron bunch. For that, the proton bunch can be sent to a four-bend chicane designed to compress it longitudinally, and then sent to a plasma channel. Assume that the energy spread of the proton beam is $\Delta E/E$ =0.01% and the initial bunch length is $\sigma_z^p = 0.2$ m.



(a) Estimate the bunch length to which the proton bunch can be compressed. Estimate the necessary plasma density n_e for the plasma acceleration channel. [6]

(b) Estimate the length of the plasma acceleration channel L_p that can accelerate the electron bunch from the initial energy of 100 GeV to 10 TeV. [6]

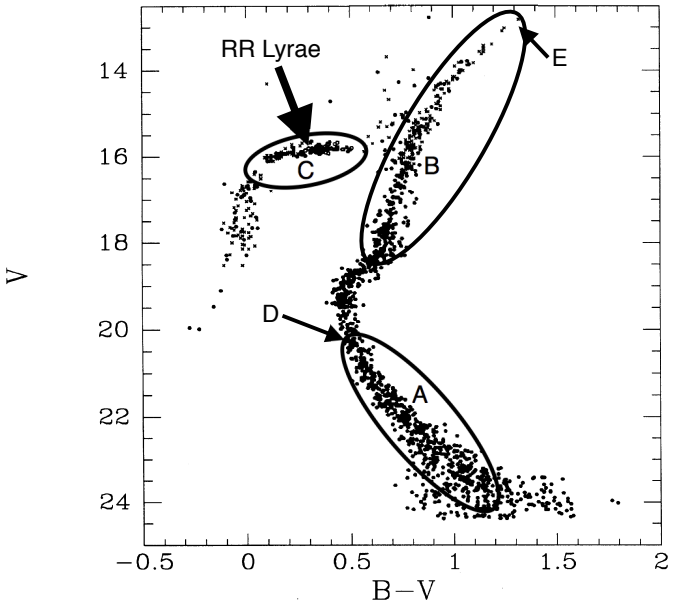
(c) Define the maximum wavelength of the laser that can be used to pre-ionize the gas, to create this plasma. [6]

(d) Considering only the effects of dephasing, estimate the number of stages, in case the acceleration of the electron bunch to this energy would be performed by laser-plasma acceleration using a laser with $\lambda = 1$ μm. Suggest the other important phenomena that can influence the number of stages. [7]

Section S26 STARS AND GALAXIES

1. Where does interstellar dust originate from, and how does this dust affect the observed brightness and colour of background stars? What properties of RR Lyrae stars make them good distance indicators? [4]

The plot below shows the Herzsprung-Russell colour–magnitude diagram for stars in the cluster M15. The large arrow indicates where RR Lyrae stars in this cluster are found, with average apparent magnitude of $V = 16.0$ in the V-band and an observed colour of $(B - V) = 0.3$ magnitudes. In the absence of dust, the average absolute magnitude in the V-band of RR Lyrae stars is $M_V = +0.6$, and their intrinsic colour is $(B - V)_0 = 0.2$ magnitudes. What is the colour excess towards M15, and what is the distance modulus to the cluster? Calculate the distance in parsecs, assuming that the extinction in magnitudes at V-band is given by $A_V = 3.1 \times E(B - V)$. [4]



In the above plot, describe where (a) hot and bright and (b) cool and faint stars are located. The colour–magnitude diagrams of clusters such as this are often supposed to be *isochrones*; describe what is meant by this. What are names of regions A, B and C, and what evolutionary stage of stars occupies each region? Describe what transitions are happening at point D (where the apparent magnitude $V = 20.3$) and at point E. [9]

For stars similar in mass to the Sun, stellar mass, M, and luminosity, L, scale as $L \propto M^4$ in region A of the plot. Estimate the mass of a star at point D on the plot. How does the lifetime of stars in region A scale with stellar mass, M? Estimate the age of this star cluster, stating any assumptions you make. Comment on the age determined in the context of the present age of our Sun and the Universe.
[The energy liberated in fusing 4 protons to form one ^{4}He nucleus is 26.7 MeV, and the absolute magnitude of the Sun in V-band is $M_V = +4.74$] [8]

2. What is meant by *hydrostatic equilibrium*? Derive the equation of hydrostatic equilibrium for the pressure gradient $\mathrm{d}P/\mathrm{d}r$ at a radius r enclosing a mass M. [3]

What is a white dwarf star? How does it shine, and how is it supported against gravitational collapse? For a white dwarf star where the electrons are non-relativistic, show that the equation of state is given by

$$P = K_1 \, (1 + X)^{5/3} \, \rho^{5/3},$$

where P is the pressure, ρ is the density, X is the mass fraction of hydrogen and K_1 is a constant to be determined.

By combining the equation of state with the equation for hydrostatic equilibrium, derive an expression for the radius of the white dwarf, R_S, in terms of the mass of the star, M, stating any assumptions you make. Hence estimate the radius of a $1\,M_\odot$ white dwarf star which has no hydrogen.

Sirius B is a white dwarf with an effective surface temperature of 25,000K, and has a luminosity of $0.056\,L_\odot$. From this, derive an independent estimate of the radius and compare your result with the previous estimate, given that Sirius B has a mass of $1\,M_\odot$ [11]

Except for the thin outer layer, justify why the interiors of white dwarf stars are approximately isothermal. Why are the nuclei non-degenerate? Assuming this is a carbon white dwarf, and that the ${}^{12}_{6}\mathrm{C}$ nuclei contain most of the thermal energy and behave as an ideal gas, find an expression for the total thermal energy of the star, U, in terms of the interior temperature, T_c, and the mass, M. [4]

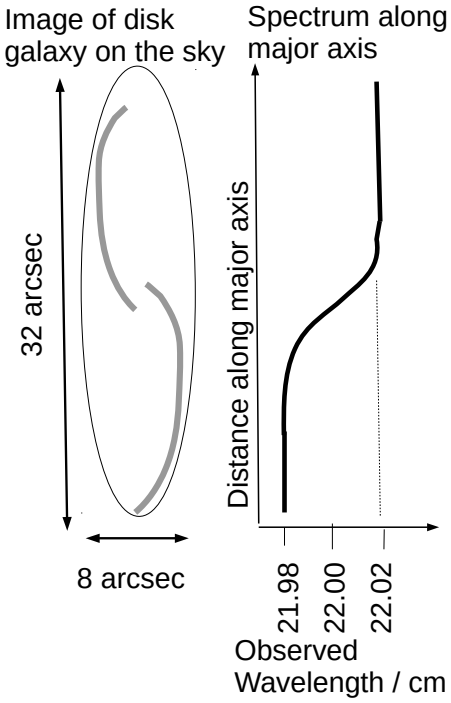
The luminosity, $L_{WD} = -\mathrm{d}U/\mathrm{d}t$, of a white dwarf is related to its core temperature, T_c, by

$$L_{WD} = 0.09 \times \frac{M}{M_\odot} \, T_c^{\frac{7}{2}} \text{ Watts.}$$

What is the current core temperature of Sirius B? How long would it take for this core temperature to drop to 10% of its current value? What would be the effective surface temperature at this time? Why do we not observe any white dwarf stars this cool? [7]

3. On a sketch of a spiral galaxy, label the key components. What are the approximate sizes of these components in the case of our galaxy, the Milky Way? Where are young stars and very old stars found? [4]

Observations of an emission line at rest-frame wavelength 21.1 cm are often use to measure the rotation of galaxies. What physical mechanism is the origin of this emission line, from where in the galaxy would you expect to see this emission, and what sort of telescopes are used to observe it? In the diagram below, a spiral galaxy is observed in this way. If the average wavelength of the same emission line for this galaxy is observed to be 22.0 cm, calculate the redshift of this galaxy and infer the distance from the observer to the galaxy (assuming the Hubble constant is $71\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$). Calculate the radius (in parsecs) of the galaxy and, assuming the disk is thin, find the inclination angle of the galaxy to the line-of-sight. What is the maximum rotation velocity? Assuming there is no matter outside the ellipse shown of major axis 32 arcseconds, calculate the total dynamical mass of this galaxy (express your answer in units of solar masses, $M_\odot$). [10]



If the total apparent magnitude of this galaxy in the visible is $m_V = 16$, calculate its luminosity in units of $L_\odot$ (you may take the absolute magnitude of the Sun to be $M_V = 4.74$ magnitudes). Compare your results for the dynamical mass of this galaxy with an estimate of the mass in stars, and comment on your result. Explain why a simple mass-traces-light model for an exponential disk cannot reproduce the observed rotation curve. [6]

Another spiral galaxy is observed with $v_{\mathrm{max}} = 100\,\mathrm{km\,s^{-1}}$. Using your results above, estimate the luminosity of this galaxy, justifying any scaling relation you use. [5]

Section S27 PHILOSOPHY OF SPACE-TIME

1. Carefully derive the Lorentz Transformations, making clear which of your assumptions are conventions and which are physical posits. [25]

2. Are Newton's First and Second Laws just definitions? [25]

3. In what sense, if any, is simultaneity relative and/or conventional? [25]

4. Can the twin paradox be satisfactorily explained by appeal to the geometry of spacetime? [25]

5. What is the distinction that Einstein introduced between "constructive" and "principle" theories, and why did he introduce it? [25]

Section S28 PHILOSOPHY OF QUANTUM MECHANICS

1. What is the measurement problem, and why if at all should physicists care about it? [25]

2. If the Everett (Many-Worlds) interpretation is correct, how are we to understand probability? [25]

3. What is the de Broglie–Bohm (Pilot Wave) theory, and how does it resolve the quantum measurement problem? [25]

4. Carefully derive Bell's inequality and explain its significance. [25]

5. Does the observed fact that systems are not in weird superpositions comprise empirical evidence for wave function collapse? [25]

Section S29 EXPLORING SOLAR SYSTEMS

1. Define the term *radiative equilibrium temperature* for a planet or asteroid. For an object in our Solar System at distance R_{AU} from the Sun (measured in astronomical units), with a Bond albedo of A and emissivity ε, show that the radiative equilibrium temperature is given by:

$$T_{eq} = \left(\frac{F_S}{R_{AU}^2} \frac{(1-A)}{4\varepsilon\sigma} \right)^{\frac{1}{4}},$$

where F_S is the solar constant. [5]

What other assumptions have you made in deriving this equation? How would the equation be modified in the absence of an atmosphere? [3]

Using the values in the table below, estimate the radiative equilibrium temperatures for the following Solar System objects; you may assume that the emissivity is 1 in all cases. [4]

Object Name	Distance (AU)	Bond Albedo
Earth	1	0.31
Moon	1	0.11
Io	5.2	0.56
Jupiter	5.2	0.34

Comment on how close to measured values these temperatures are for (a) the surface temperature of the Moon (mean equatorial temperature of $\approx 206\,\mathrm{K}$) and (b) the emission temperature of Jupiter's atmosphere ($\approx 130\,\mathrm{K}$). [4]

With reference to the images below, explain if we believe that the surfaces of these moons are likely to be geologically active or not. Comment on the likely sources of energy required to drive any dynamic re-surfacing, given the radiative equilibrium temperature you have estimated earlier. [9]

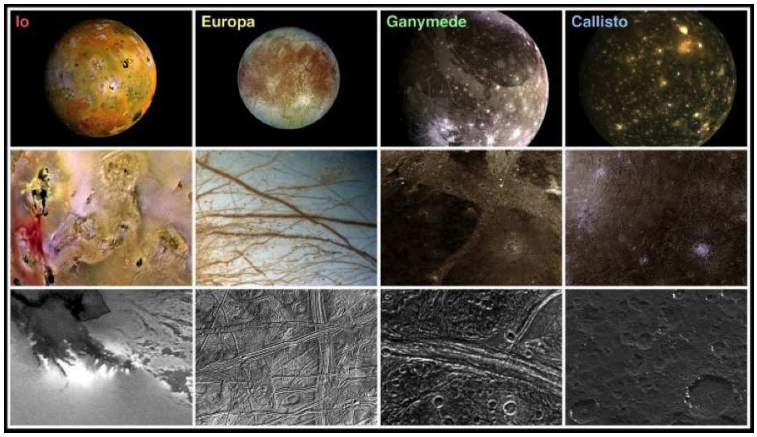


Image: Galilean satellites of Jupiter, NASA/JPL/DLR

2. Qualitatively describe the three key stages of impact crater formation. [6]

In the case of an airless body, such as the Earth's Moon, the ejecta produced by an impact follow near ballistic trajectories. Stating any assumptions, show that the ejecta from a primary impact will re-impact at a distance of

$$r = \frac{v_{ej}^2}{g_p} \sin 2\theta,$$

where r is the re-impact distance from the impact point, g_p is the local acceleration due to gravity for the planet, v_{ej} is the velocity of the ejecta and θ is the angle the ejecta makes to the local horizontal. State the angle at which this range is a maximum. [9]

Give a qualitative justification for why the diameter of a crater, D, is roughly proportional to the impact energy to the $\frac{1}{4}$ power. [2]

In 2009, NASA's LCROSS mission impacted the upper stage of a rocket into one of the permanently shadowed craters at the south pole of the Moon. The rocket's upper stage was travelling at $2.5\,\mathrm{km\,s^{-1}}$ normal to the lunar surface. It is estimated that the impact excavated a crater $\approx 30\,\mathrm{m}$ in diameter. Using this information and assuming that the mechanical properties of the regolith at the LCROSS impact site are broadly representative for the Moon as a whole, estimate the maximum distance from the impact site at which you would expect to find a secondary crater due to an impactor of a similar mass to one that formed a 75 m diameter circular crater. State any other assumptions you make. You may assume that 5% of the velocity from the impact is typically transferred to the secondary crater-forming ejecta. [6]

Why was the LCROSS mission specifically targeted to a permanently shadowed polar crater? [2]

[The acceleration due to gravity at the Moon's surface is $1.622\,\mathrm{m\,s^{-2}}$.]

3. By measuring the polar moment of inertia, C, of a planet it is possible to gain an initial insight into the mass distribution within the planet. For the following mass distributions, show that the moment of inertia, C, is:

(a) a hollow sphere with an outer layer of uniform density, $C = \frac{2}{3}MR^2$

(b) a sphere of uniform density, $C = \frac{2}{5}MR^2$

(c) a sphere with all of the mass concentrated at the centre, $C = 0$

where M and R are the mass of the planet and its equatorial radius respectively. [9]

In the case of a giant planet such as Jupiter, what other measurements in addition to the mass and the radius are required for an accurate determination of the polar moment of inertia? [2]

Moment of inertia data for the Earth, Jupiter and Saturn are given in the table below. Based on the moments of inertia you have calculated previously, what does this imply for the internal mass distributions for these three planets? [5]

.

Planet	$\frac{C}{MR^2}$
Earth	0.331
Jupiter	0.264
Saturn	0.21

Draw a qualitative sketch of the possible internal structure of Jupiter and Saturn, explaining where they are similar and where they are different (exact dimensions and pressures are not required). [5]

Measurements of the hydrogen/helium ratio for Jupiter and Saturn show that Jupiter's is similar to the Sun's but that Saturn's is lower. Observations of Saturn's emission temperature show that the planet is emitting more energy than can be explained by primordial heat release alone. Outline why these two observations (of H/He ratio and emission temperature) may indicate an important difference in behaviour of He deep in Saturn's interior compared with Jupiter. [4]