A10289W1

FIRST PUBLIC EXAMINATION

Long Vacation

Preliminary Examination in Physics

Paper CP4: MATHEMATICAL METHODS 2

also

Preliminary Examination in Physics and Philosophy

Thursday 8 September 2016, 9.30 am – 12 noon

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a fresh page.

The use of calculators is permitted.

A list of physical constants, mathematical formulae and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ for the following cases

(a) $y = x^{\tan x}$

(b) $2y = 5x^3 + \sin y$. [2]

2. Given the integral

$$\int_{-1}^{0}\int_{x}^{x^2} y \,\mathrm{d}y \,\mathrm{d}x$$

(i) sketch the region of integration, (ii) write an equivalent integral with the order of integration reversed and (iii) evaluate the integral both ways. [7]

3. Evaluate the following for the case where $\mathbf{a}$ is a constant vector, $\mathbf{r}$ is the position vector and $r = |\mathbf{r}|$

(i) $\nabla(\mathbf{a} \cdot \mathbf{r})$
(ii) $\nabla \cdot (\mathbf{a} \times \mathbf{r})$
(iii) $\nabla \cdot (r^{-3}\mathbf{r})$ [6]

4. Generally speaking, when are phase and group velocities equal? Surface waves on a liquid of surface tension T and density ρ have a phase velocity

$$v_p = \left(\frac{g\lambda}{2\pi} + \frac{2\pi T}{\lambda\rho}\right)^{1/2}$$

where λ is the wavelength of the wave and g is the gravitational acceleration. Find the group velocity of the surface waves in terms of λ, g, T, and ρ. [4]

5. Suppose $u = u(x, y)$ and $y = y(x)$. Find a formula for $\frac{\mathrm{d}u}{\mathrm{d}x}$ in terms of partial derivatives of u. Use your formula to find $\frac{\mathrm{d}u}{\mathrm{d}x}$ for the case where $u = 3x^2 + y^2$ and $y = \sin x$. [5]

6. Show that the line integral

$$\oint_C \mathbf{v} \cdot \mathrm{d}\mathbf{l}$$

around any closed loop C is zero for the case of the vector $\mathbf{v} = 2y\mathbf{i} + 2(x + yz)\mathbf{j} + y^2\mathbf{k}$. [4]

7. Find the first three non-zero terms of the Maclaurin series for

$$\ln\frac{1+x}{1-x}$$

. [4]

8. Demonstrate the validity of the divergence theorem by calculating the surface integral $\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$ over the closed surface defined by $z = 3 - x^2 - y^2$ and the plane $z = 0$ where $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$. [8]

Section B

9. (a) Outline the steps needed to transform an integral $\iint_R f(x,y)\mathrm{d}x\,\mathrm{d}y$ written in terms of the integration variables (x,y) into an integral in terms of different integration variables (u,v) where u and v are each functions of x and y and R is the region of integration. [3]

(b) Use the transformation $u = y/x$, $v = xy$ to evaluate

$$\iint_R x^3 y \mathrm{d}x\,\mathrm{d}y$$

over the region R in the positive xy quadrant enclosed by $y = x$, $y = 2x$, $xy = 1$, and $xy = 3$. Explain, with the aid of sketches, how the limits of integration were chosen. [8]

(c) By making an appropriate change of variables, evaluate the integral

$$\iint_R \left(\frac{x-y}{x+y+3}\right)^2 \mathrm{d}x\,\mathrm{d}y$$

where R is the region bounded by the lines $x + y = 1$, $x + y = 3$, $x - y = 0$, $x - y = 1$. [9]

10. (a) Derive the expression $P = c\rho A^2\omega^2/2$ for the power carried by a wave travelling at speed c with amplitude A and angular frequency ω on a string with constant tension T and mass per unit length ρ. [6]

(b) Calculate the power in Watts carried by a travelling wave of amplitude 5 mm and wavelength 0.5 m on an infinitely long string of mass per unit length 0.1 kg m^{-1} and a tension of 250 N. [3]

(c) A composite string comprises a string of mass per unit length 0.1 kg m^{-1} joined at $x = 0$ to a heavier string of mass per unit length 0.9 kg m^{-1}. The composite string is held under constant tension. A sinusoidal wave of amplitude A travels on the lighter string towards the heavier string.

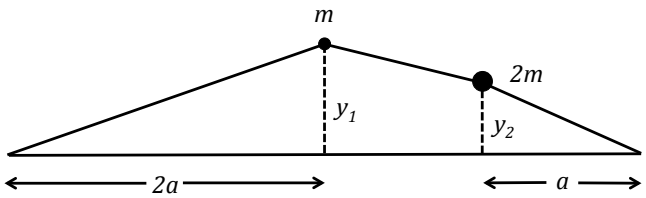
(i) Show that the ratio of the amplitudes of the transmitted to incident wave is

$$t = \frac{2k_1}{k_1 + k_2}$$

where k_1 and k_2 are the wave numbers of the waves on the lighter and heavier strings respectively. [6]

(ii) Assuming the amplitude of the incident wave is 5mm and its wavelength is 0.5m, what fraction of the power carried by the incident wave is transmitted to the heavier string? [5]

11. The ends of a light elastic string under tension T are fixed at a distance $4a$ apart. A mass m is attached a distance $2a$ from one end of the string, and a mass $2m$ is attached a distance a from the other end.



(a) Ignoring gravity, show that the equations of motion describing small transverse displacements, y_1 and y_2, of the masses are given by

$$\frac{d^2y_1}{dt^2} + \frac{T}{2ma}(3y_1 - 2y_2) = 0$$

$$\frac{d^2y_2}{dt^2} + \frac{T}{2ma}(2y_2 - y_1) = 0$$

[4]

(b) Find the frequencies of oscillations of the normal modes. [4]

(c) Find and describe the relative displacements of the two masses in each of the normal modes. [4]

(d) Find the general solution for y_1 and y_2 at time t. [3]

(e) The masses are released from rest with the initial displacements $y_1 = d$ and $y_2 = 0$. Find expressions for the subsequent displacement of each of the two masses as a function of time. [5]

12. A continuous surface is defined by the function $z = f(x, y)$. Write down the two-dimensional Taylor expansion about point (a, b) up to quadratic terms. [2]

Explain the meaning of $\left(\frac{\partial f}{\partial x}\right)_y$. Why do the conditions $\left(\frac{\partial f}{\partial x}\right)_y = 0$ and $\left(\frac{\partial f}{\partial y}\right)_x = 0$ ensure that a function is locally flat? [5]

Write an approximate expression for the difference in z between that found at a stationary point and a nearby point. Explain how this approximate expression may be used to determine the nature of stationary points and derive the conditions for a maximum, minimum and a saddle point. [6]

Find the position and nature of all the stationary points of the function

$$f(x, y) = x^2 + y^2 + 3x^4 - 3y^4$$

and sketch the contours of constant f. [7]