A10289W1

FIRST PUBLIC EXAMINATION

Trinity Term

Preliminary Examination in Physics

Paper CP4: MATHEMATICAL METHODS 2

also

Preliminary Examination in Physics and Philosophy

Thursday 9 June 2016, 2.30 pm – 5 pm

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a new page.

The use of approved calculators is permitted.

A list of physical constants, mathematical formulae and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. Find the arc length of the curve given by

$$x(t) = a(\cos t + t \sin t)$$

$$y(t) = a(\sin t - t \cos t)$$

from $t = 0$ to 1 where a is a constant. [5]

2. Evaluate the following limits:

(a)

$$\lim_{x \to 1} \frac{2^x - 2}{x - 1}$$

[2]

(b)

$$\lim_{x \to 0} (\ln(1 + x))^x$$

[3]

3. Suppose $xy^2 + yz^2 = 2$ and $v = xy^3$ where x, y, z and v are all variables. Using a chain rule, express $(\partial v/\partial z)_x$ as a function of x, y, z and evaluate it for $(x, y, z) = (1, 1, 1)$. [7]

4. For $\mathbf{F} = x^3\mathbf{i} + (y^3 + 2xy)\mathbf{j} - 2xz\mathbf{k}$, using any valid method, find the flux of the vector field $\mathbf{F}$, (i.e. $\int \mathbf{F} \cdot d\mathbf{S}$), through the closed surface formed by a paraboloid $z + x^2 + y^2 \leq 1$ and $z \geq 0$ from which the positive xy quadrant has been removed. [6]

5. Use a Taylor expansion to find a linear approximation to $f(x, y) = x^2 e^y$ about the point (x_0, y_0). Hence estimate the value of f at the point $(2.03, -0.01)$. In terms of significant digits, how accurate is your approximation? [5]

6. Determine whether $\mathbf{F} = 2xz\mathbf{i} + y\mathbf{j} + x^2\mathbf{k}$ is conservative. Evaluate the line integral

$$\int_C \mathbf{F} \cdot \mathbf{dr}$$

along the curve C represented by the parametric equations

$$x(t) = (2 + \cos(5t)) \cos t$$

$$y(t) = (2 + \cos(5t)) \sin t$$

$$z(t) = \sin(5t)$$

from $t = 0$ to $t = \pi/2$. [5]

7. Find the positions and nature of the stationary points of the function $f(x, y) = x^3 + y^3 - 2x - 2y$. Make a rough sketch of the contours of constant f. [7]

Section B

8. (a) Write down a mathematical expression for Stokes' theorem for the vector field $\mathbf{F}$ and carefully describe the theorem in words. [3]

(b) For the case $\mathbf{F} = \psi\mathbf{B}$ where ψ is an arbitrary scalar function and $\mathbf{B}$ is an arbitrary constant vector, derive the following vector integral theorem

$$\oint_C \psi \, \mathrm{d}\mathbf{r} = \iint_S (\hat{\mathbf{n}} \times \nabla\psi) \, \mathrm{d}S$$

where C is the curve bounding the surface S and $\hat{\mathbf{n}}$ is a unit vector normal to the surface S. [6]

(c) For the case where

$$\mathbf{V} = (xy + e^x)\mathbf{i} + (x^2z - 3y)\mathbf{j} + (y^2 + z^2)\mathbf{k}$$

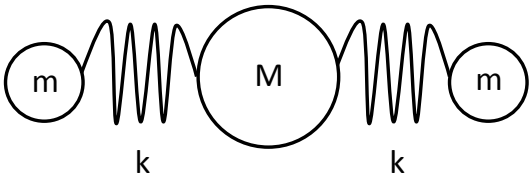
and $\hat{\mathbf{n}}$ is a unit vector normal to the surface S, find

$$\iint (\nabla \times \mathbf{V}) \cdot \hat{\mathbf{n}} \, \mathrm{d}S$$

over any surface whose bounding curve is in the xz plane. [4]

(d) Given the vector field $\mathbf{A} = x^2z\mathbf{i} + xy\mathbf{j} - 3yz\mathbf{k}$, verify Stokes' theorem for the case where the bounding curve is obtained by intersecting the plane $x = -z + 1$ with the unit cube bounded by the coordinate planes and the planes $x = 1$, $y = 1$, $z = 1$. [7]

9.



A simple model for a linear triatomic molecule is three atoms connected by two springs. Assume that the atoms are constrained to move in one dimension, and that the springs are identical with spring constant k. Also assume that the central atom, with mass M, is more massive than the other two, which we assume to have equal masses, m.

(a) Considering the atoms in the triatomic molecule from left to right, define their displacements from their equilibrium positions as x_1, x_2, x_3. Find an equation of motion for each of the three masses. [3]

(b) Explain in words what is meant by a normal mode of a system. How many normal modes exist for the linear triatomic molecule? [2]

(c) Obtain expressions for the frequencies of the normal modes of the simple model of the triatomic molecule described above. [5]

(d) Find the relative motions and sketch the displacements as a function of time of each of the three masses in each of the normal modes. [4]

(e) For the case where $M = 2m$ and the atoms are released from rest with initial displacements from their equilibrium positions of $x_1 = 0$, $x_2 = d$ and $x_3 = -d$, find expressions for the subsequent displacement of each of the three masses as a function of time. [6]

10. (a) Consider the surface S described by the equation $z = f(x, y)$. Derive the following expression for the area of this surface

$$\iint_S \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, \mathrm{d}x\, \mathrm{d}y$$

[6]

(b) Evaluate the following two surface integrals:

(i)

$$\iint x^2\, y^2\, \mathrm{d}S$$

over the curved surface given by the part of the cone $x = \sqrt{y^2 + z^2}$ that lies between the planes $x = 2$ and $x = 3$; [7]

(ii)

$$\iint \mathbf{A} \cdot \mathrm{d}\mathbf{S}$$

over the portion of the plane $y = 3 - x - 2z$ in the positive octant (i.e. $x \geq 0$, $y \geq 0$, $z \geq 0$) where $\mathbf{A} = x\mathbf{i} - x\mathbf{j} + z\mathbf{k}$. [7]

11. A uniform string of length L, mass per unit length ρ and tension T is fixed at all times at both ends $x = 0$ and $x = L$. Initially the string is held taut between its ends and is struck so that it instantaneously acquires a transverse velocity profile $V(x)$. As a result it starts vibrating according to the wave equation

$$\frac{\partial^2 y(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2} = 0\,,$$

where $y(x,t)$ gives the shape of the string and c is a non-zero real constant which depends on the properties of the string according to $c = \sqrt{\frac{T}{\rho}}$.

(a) Assuming that $y(x,t) = f(x)g(t)$, where $f(x)$ and $g(t)$ are non-zero, show that there exists a constant α such that the following two relationships hold

$$f'' = \alpha f$$

$$g'' = \alpha\, c^2\, g$$

[2]

(b) Given the boundary conditions and that we are only interested in solutions where $y(x,t)$ exhibits wave behaviour, find the possible values of α. [3]

(c) For the case where $V(x) = V_0(\frac{1}{2}\sin(\pi x/L) - \frac{3}{4}\sin(2\pi x/L))$ at $t = 0$ and V_0 is a constant with dimensions of velocity, find the solution for $y(x,t)$ and hence expressions for the time dependence of the displacement amplitudes of the modes on the string. [6]

(d) Show explicitly how each mode in the solution for $y(x,t)$ relates to travelling waves. [3]

(e) By calculating the potential and kinetic energy of the string, deduce which mode contributes most to the total energy at any given time t. [6]