A10285W1

FIRST PUBLIC EXAMINATION

Long Vacation

Preliminary Examination in Physics

Paper CP3: MATHEMATICAL METHODS 1

also

Preliminary Examinations in Physics and Philosophy

Wednesday 7th September 2016, 9:30am – 12:00 noon

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

The use of calculators is **not** *permitted.*

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. Sketch on an Argand diagram the locus of complex numbers z such that
(i) $z^3 = 27$
(ii) $(z-2+i)^2 + (z^*-2-i)^2 + 2zz^* + 6 = 0$, where z^* is the complex conjugate of z. [6]

2. Reduce the determinant $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$ to a product of two factors, each symmetric in a, b, c. [4]

3. The equations of two spheres are

$$\begin{aligned} (\mathbf{r}-\mathbf{a})\cdot(\mathbf{r}-\mathbf{a}) &= \alpha^2 \\ (\mathbf{r}-\mathbf{b})\cdot(\mathbf{r}-\mathbf{b}) &= \beta^2 \end{aligned}$$

where $\mathbf{r}$ is a position vector, $\mathbf{a}$ and $\mathbf{b}$ are constant vectors, and α, β are constants. Find the condition for the spheres to intersect. For the case where the spheres do intersect, show that the position vector of the centre of the circle of intersection is given by

$$\frac{1}{2}\left(\mathbf{a}+\mathbf{b}+\frac{(\alpha^2-\beta^2)(\mathbf{b}-\mathbf{a})}{|\mathbf{b}-\mathbf{a}|^2}\right)$$

Find the condition satisfied by α, β and $|\mathbf{b}-\mathbf{a}|$ when the centre of the circle of intersection lies at the centre of the first sphere, and interpret your result geometrically. [8]

4. A cylinder of radius r is rolling down a plane which is inclined at angle α to the horizontal. The axis of the cylinder moves at constant velocity v. Let (x, y) be the horizonal and vertical coordinates of a point on the cylinder at distance a from the cylinder's axis, and set $t = 0$ at a time when this point is directly above the axis. Obtain equations for $x(t)$ and $y(t)$. Hence, or otherwise, determine the condition on a that the point sometimes moves in an exactly upwards direction. [5]

5. Find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + xy = x + y - 1$$

and plot a graph of the solution $y(x)$ as a function of x for the case where $y = 2$ at $x = 1$. [8]

6. A plane contains the points $(1, 2, 3)$, $(4, 5, 6)$ and $(\gamma, 8, 7)$ for some constant γ. Prove that there is a value of γ for which the plane passes through the origin, and find that value. [3]

7. Find all 2×2 diagonal matrices A that satisfy the equation $A^2 + 2A = 15I$, where I is the identity matrix. Also find a matrix of the form $A = aI + bX$ that satisfies this equation, where $a, b \neq 0$ are constants to be discovered, and $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. [6]

Section B

8. (a) Show that the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + f(x)y = g(x)y^m$$

can be reduced to the form

$$\frac{\mathrm{d}z}{\mathrm{d}x} + h(x)z = k(x)$$

by the substitution $z = y^\alpha$ for a suitable choice of α. [4]

Find the general solution of the following differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\tan x = y^2\tan^2 x$$

[6]

(b) Using the transformation $y(x) = x^2 + \dfrac{1}{v(x)}$, find the general solution of the differential equation

$$2x^3\frac{\mathrm{d}y}{\mathrm{d}x} = 2y^2 + yx^2 + x^4$$

Use your solution to find the value of $y(4)\,(3 - 2y(1))$. You may assume the solution is not singular at $x = 1$ or $x = 4$. [10]

9. (i) State the relationship between $\mathbf{u}^{\mathrm{T}}\mathbf{v}$ and $\mathbf{v}^{\mathrm{T}}\mathbf{u}$ for arbitrary column vectors $\mathbf{u}$, $\mathbf{v}$.
(ii) For the case where

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix},$$

show that $\det(\mathrm{A} - \mathrm{I}) = 0$, where I is the identity matrix, and find the eigenvalues and eigenvectors of A. [5]

(iii) A surface in three dimensions is given by the equation $\mathbf{r}^{\mathrm{T}}A\mathbf{r} + \mathbf{b}^{\mathrm{T}}\mathbf{r} = 1$ where $\mathbf{b} = (4, 0, 2)^{\mathrm{T}}$ is a fixed vector and $\mathbf{r} = (x, y, z)^{\mathrm{T}}$ is the general position vector. By writing $\mathbf{r} = \mathbf{x} + \mathbf{a}$ where $\mathbf{a}$ is a constant vector to be discovered, find a condition relating $\mathbf{a}$ to A and $\mathbf{b}$ such that the equation for the surface has the form $\mathbf{x}^{\mathrm{T}}A\mathbf{x} = C$ where C is a constant. Hence determine the constant vector $\mathbf{a}$ and the constant C. [6]

(iv) By considering a rotation of the axes, find, without determining the rotation matrix, the nearest distance from the point $\mathbf{a}$ to the surface, and the maximum value of y on the surface. [5]

(v) Consider in turn the effect of replacing the matrix A in the above by

$$A' = \begin{pmatrix} 2 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \quad \text{and} \quad A'' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{pmatrix}.$$

In each case, provide the answers to the questions posed in part (iv). [4]

10. For the case where

$$A = \begin{pmatrix} 4 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix},$$

confirm that one of the eigenvalues of A is equal to 1.
Find a matrix V such that $D \equiv V^T AV$ is a diagonal matrix and $V^{\mathrm{T}}V = I$. Write down an expression for A in terms of D and V. [6]

The exponential function applied to an arbitrary square matrix M is defined by

$$\exp M = \sum_{n=0}^{\infty} \frac{M^n}{n!}\,.$$

Let

$$B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}.$$

Find $\exp B$, $\exp C$ and the diagonal elements of $\exp A$. [11]

For ordinary numbers a, b, we have that $\mathrm{e}^a\mathrm{e}^b = \mathrm{e}^{a+b}$. By writing this in terms of a pair of series, or otherwise, prove that for a pair of square matrices A, B, one has $\exp(A)\exp(B) = \exp(A+B)$ when A commutes with B. [3]

11. A body of mass m moving in one dimension is subject to a force

$$f = -k(x - x_0) - \gamma m v$$

where x is the position of the body, v is its velocity and k, x_0, γ are constants. Describe the physical significance of these constants, and the qualitative type of motion expected when γ is small. [4]

The body is initially held still at $x = a$ and then released at $t = 0$. Find the subsequent motion $x(t)$, assuming that $\gamma^2 < 4k/m$. [6]

(i) In the limit of small damping, find the number of oscillations required for the amplitude of the oscillations to fall to $\exp(-1/2)$ times its initial value.

(ii) In the same limit, find the ratio of the energy stored to the energy lost per cycle.

(iii) Suppose now that the location x_0 is made to vary as

$$x_0 = de^{-\alpha t}$$

where d and α are constants, and the initial conditions are now $x = d$, $\dot{x} = -\alpha d$ at $t = 0$. Find the new expression for $x(t)$ (once you have an expression in terms of the given constants and time, you are not required to simplify it for general values of the constants). Simplify the expression in the case $\alpha = \gamma$, and hence describe the motion of the body in that case. [10]