A10285W1

FIRST PUBLIC EXAMINATION

Trinity Term

Preliminary Examination in Physics

Paper CP3: MATHEMATICAL METHODS 1

also

Preliminary Examinations in Physics and Philosophy

Wednesday 8th June 2016, 2:30pm – 5:00pm

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

The use of calculators is **not** *permitted.*

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. Sketch on an Argand diagram the locus of complex numbers z such that
(i) $\sin(|z|) = 0$, for $|z| < 10$
(ii) $\mathrm{Re}\{\log z\} = 2$ [4]

2. Find the real and imaginary parts of $\sin(2+3i)$, expressing each part as a product of hyperbolic and trigonometric functions. [4]

3. Let $\hat{\mathbf{n}}$ be a unit vector, and let d be a constant. What is the locus of points $\mathbf{r}$ such that $\mathbf{r} \cdot \hat{\mathbf{n}} = d$? [2]

4. $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three non-coplanar vectors. Vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are defined by

$$\mathbf{A} = \frac{\mathbf{b} \times \mathbf{c}}{\Delta}, \quad \mathbf{B} = \frac{\mathbf{c} \times \mathbf{a}}{\Delta}, \quad \mathbf{C} = \frac{\mathbf{a} \times \mathbf{b}}{\Delta},$$

where $\Delta = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. Show that the vector

$$\mathbf{X} = \alpha\mathbf{A} + \beta\mathbf{B} + \gamma\mathbf{C}$$

is normal to the plane P containing the three points $\mathbf{a}/\alpha$, $\mathbf{b}/\beta$, $\mathbf{c}/\gamma$. [6]

5. For a given constant λ, show that the system of linear equations

$$\begin{aligned} 2x - 2y + z &= \lambda x \\ 2x - 3y + 2z &= \lambda y \\ -x + 2y &= \lambda z \end{aligned}$$

has a non-trivial solution only if λ has one of two values. Find these values, and give the general solution of the equations in the two cases. Interpret your results geometrically when x, y, z are Cartesian coordinates. [9]

6. Find the perpendicular distance of the point $(1, 2, 3)$ from the straight line that passes through $(6, 5, 4)$ and $(3, 3, 3)$. [6]

7. In a simple model, a car may be treated as a point mass M above a single small wheel which rests on the road surface. The mass is joined to the axle of the wheel by a spring of constant k in parallel with a viscous damper of rate γ. The car travels at constant speed v along a long straight bumpy road. The height y_0 of the road surface is given by $y_0 = B \sin x$, where x is the distance along the road and B is a constant. Find an expression for the amplitude A of the vertical motion of the car, as a function of v, after transient behaviour has died away. Sketch a graph of $A(v)$. Explain what problems arise for travellers in the car if γ is (i) very small or (ii) very large. [9]

Section B

8. The quadratic forms $Q_1(x, y, z)$ and $Q_2(x, y, z)$ are defined by

$$\begin{aligned} Q_1(x, y, z) &\equiv x^2 - 4y^2 + z^2 + 4yz - 4zx - 4xy, \\ Q_2(x, y, z) &\equiv x^2 + y^2 + z^2. \end{aligned}$$

Write down a real symmetric matrix A such that $Q_1 = \mathbf{r}^{\mathrm{T}} A \mathbf{r}$, where $\mathbf{r}$ is a column vector with components x, y, z. Show that the characteristic polynomial of A is

$$\lambda^3 + c\lambda^2 - 19\lambda - 20$$

and determine the constant c. [6]

Find a real orthogonal transformation P, such that $\mathbf{r} = P\mathbf{R}$, where $\mathbf{R} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$, which simultaneously takes $Q_1(x, y, z)$ into

$$\tilde{Q}_1(X, Y, Z) = -X^2 + \beta Y^2 + \gamma Z^2$$

and $Q_2(x, y, z)$ into $\tilde{Q}_2(X, Y, Z) = X^2 + Y^2 + Z^2$ respectively, and determine the real numbers β, γ, where $\beta < 0$ and $\gamma > 0$. [6]

Let Σ be the surface defined by the equation $Q_1(x, y, z) = -5$. Describe in full the shape of the intersection of Σ with the plane through the origin perpendicular to the direction $(2, 1, -2)$. Also provide some descriptive words and a very rough sketch of Σ, sufficient to indicate the overall three-dimensional shape in a qualitative fashion. [8]

9. (a) Find the general solution of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + 3$$

[3]

(b) Solve the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} \ln x + \frac{2y}{x} = 1$$

given that $y = 0$ when $x = e$. [6]

(c) Solve the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + 5y = xe^{-x}$$

given that $y = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$. [8]

(d) Find the general solution of the equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x^2}$$

[3]

10. Given that

$$A\cos x + B\sin x = R\sin(x+\phi),$$

find expressions for R and ϕ in terms of A and B. [2]

Using the substitution $z = fg$, or otherwise, solve the equations

$$\begin{aligned} f\frac{\mathrm{d}g}{\mathrm{d}t} + g\frac{\mathrm{d}f}{\mathrm{d}t} &= -3fg \\ \frac{\mathrm{d}^2 f}{\mathrm{d}t^2} &= -3f + 6fg \end{aligned}$$

so as to obtain an expression for the function $f(t)$, given that, at $t = 0$, $f = 2$, $g = 1$ and $\mathrm{d}f/\mathrm{d}t = 0$. [8]

Sketch a graph of $f(t)$ in the range $0 < t < 3$. [3]

The matrix $M(t)$ is given by

$$M = \frac{1}{2}\begin{pmatrix} f/a & -\sqrt{3} & 0 \\ \sqrt{3} & (2-g)a/4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

where a is a constant. Find the smallest positive time t at which M has determinant equal to 1. Find the value of a for which M is then a rotation matrix, and find the axis and angle of the rotation. [7]

11. (a)
(i) Write down an expression for $(AB)^{\mathrm{T}}$ in terms of A^{T} and B^{T}, where A and B are square matrices. Prove your result for the case of 2×2 matrices.
(ii) Show that, if matrices U and V are unitary, then UV is unitary.
(iii) Show that, if H is Hermitian, then so is any positive integer power of H.
(iv) If A and B are Hermitian, find the condition that (AB) is Hermitian. [8]

(b) The time evolution of a vector $\mathbf{v}$ is given by

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -iH\mathbf{v}$$

where H is a constant square matrix. For small times t, the solution can be written as a power series expansion,

$$\mathbf{v}(t) = \mathbf{v}_0 + t\mathbf{v}_1 + t^2\mathbf{v}_2 + \cdots + t^n\mathbf{v}_n + \cdots$$

where the vectors $\mathbf{v}_n$ are independent of time. Show that $\mathbf{v}_0 = \mathbf{v}(0)$ and that

$$\mathbf{v}_n = \frac{(-iH)^n}{n!}\mathbf{v}_0.$$

[6]

Given that $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, find the matrix $U(t)$ such that

$$\mathbf{v}(t) = U\mathbf{v}_0$$

[6]