A10289W1

FIRST PUBLIC EXAMINATION

Trinity Term

Preliminary Examination in Physics

Paper CP4: MATHEMATICAL METHODS 2

also

Preliminary Examination in Physics and Philosophy

Thursday 11 June 2015, 2.30 pm – 5 pm

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

The use of calculators is permitted.

A list of physical constants, mathematical formulae and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. State Gauss' theorem and use it to evaluate

$$I = \oint_S \mathbf{A} \cdot \hat{\mathbf{n}} \, \mathrm{d}S \,,$$

where $\mathbf{A} = 4xz\hat{\mathbf{i}} - y^2\hat{\mathbf{j}} + yz\hat{\mathbf{k}}$ and $\hat{\mathbf{n}}$ is a unit vector locally perpendicular to the surface elements $\mathrm{d}S$ of the cube that is bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$. [4]

2. Evaluate the limits

(a) $\lim_{x \to 0} \left[\dfrac{\sin x - x}{x^3} \right]$,

(b) $\lim_{x \to 0} \left[\dfrac{1}{x} - \dfrac{1}{e^x - 1} \right]$. [5]

3. Use series expansions for $\sin x$ and $\cos x$ to obtain the first three non-zero terms of the series expansion for $\tan x$. Hence find the first three non-zero terms of the series expansion for $\exp(\tan x)$. [5]

4. A central force field in three-dimensional space may be written as

$$\mathbf{F}(r) = \mathbf{r} f(r)$$

where $\mathbf{r}$ is a radial vector and where $f(r)$ is a function only of radius r. Obtain an expression for $\nabla \cdot \mathbf{F}$ in terms of $f(r)$ and its derivative $\frac{\mathrm{d}f}{\mathrm{d}r}$.

Hence calculate $\nabla \cdot \mathbf{F}$ for the case $f(r) = r^{n-1}$ and discuss what happens to the divergence for the case $n = -2$. [6]

5. Define the *phase velocity* and *group velocity* of a sinusoidal wave in terms of its angular frequency ω and wavenumber k. The phase velocity of light in glass increases with wavelength: show whether the group velocity is greater than or less than the phase velocity. [4]

6. Evaluate the integral

$$I = \int\int_A x \, \mathrm{d}x \, \mathrm{d}y,$$

where A is the finite area enclosed by the parabola $y = x^2$ and the straight line $2x - y + 8 = 0$. [4]

7. Sketch the region of integration in the (x, y) plane for the integral

$$I = \int_0^1 \mathrm{d}x \int_0^{\sqrt{(1-x^2)}} e^{-x^2-y^2} \mathrm{d}y$$

and evaluate I. [6]

8. Make the change of variables $r = x + vt$ and $s = x - vt$ in the wave equation

$$\frac{\partial^2 F}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 F}{\partial t^2} = 0\,,$$

and hence obtain the solution for the wave equation in terms of functions of r and s. [6]

Section B

9. Consider the function of two variables,

$$z(x, y) = f(x)g(x, y)\,,$$

where $f(x) = \cos x$ and $g(x, y) = \exp(-x^2 - y^2)$.

(a) Show that there are multiple stationary points of the function, that these lie along the x-axis and that their positions satisfy an equation of the form $\tan x = k(x)$. Give the expression for $k(x)$. [5]

(b) Write down the general expression for a Taylor series expansion of a function of two variables, to second order. Expand the function $z(x, y)$, given above, as a Taylor series in (x, y) about a general stationary point to second order, writing the value of z at a stationary point $(x_\mathrm{s}, 0)$ as z_s. [3]

(c) State the general criteria for determining the nature of the stationary values of functions of two variables. [4]

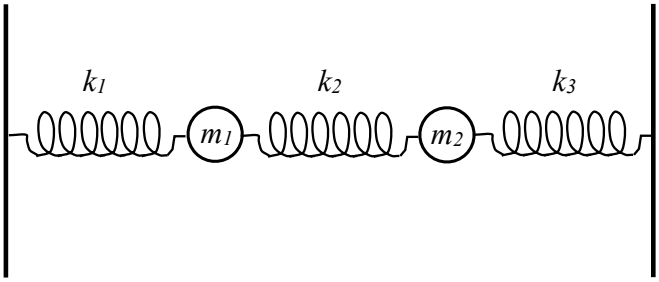
(d) Use those criteria to evaluate the nature of all the stationary points of the function $z(x, y)$. [4]

(e) Show that the function

$$w(x, y) = h(x)g(x, y)\,,$$

where $h(x) = x^3$ and $g(x, y) = \exp(-x^2 - y^2)$, has a stationary value at the origin whose nature cannot be determined by only considering derivatives up to second order. [4]

10. (a) Two masses, of mass m_1 and m_2, are connected together by a spring, and two further springs connect each mass to a fixed point, as shown in the figure below. The masses and springs are aligned horizontally, along the x-axis. The masses are free to move, free from friction and subject only to the forces imparted by the springs, along the x-axis only. Each spring may be considered to have negligible mass, and they have spring constants k_1, k_2 and k_3 as indicated in the figure.

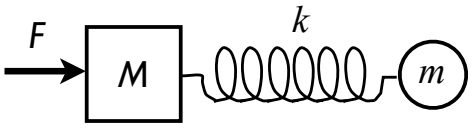


(i) Obtain an expression for the angular frequencies of the normal modes of the system, assuming k_1, k_2 and k_3 are all different. [5]

(ii) Give a simpler form for these angular frequencies for the case when both $k_1 = k_3$ and $m_1 = m_2$. [2]

(iii) Find the ratio of the amplitude of the motion of mass m_1 to that of mass m_2 for each of the normal modes in part (ii) above. Describe how the masses move for each normal mode. [6]

(b) A small mass m is connected to a larger mass M by a spring of spring constant k as shown in the figure below. The larger mass is subject to an oscillating force $F = F_0 \cos \omega t$ along the x-axis, where ω is the angular frequency, t is time, and F_0 is a constant. The masses are free to move along the x-axis only and are subject only to the force F and the force imparted by the spring.



Show that the amplitude of the forced oscillations of mass M may be made small by a suitable choice for the value of k. By considering the dependence of the amplitude of the forced oscillations on the value of k, discuss what would limit the usefulness of this result for the damping of forced oscillations? [7]

11. (a) Derive the wave equation

$$\frac{\partial^2 y(x,t)}{\partial t^2} - c^2 \frac{\partial^2 y(x,t)}{\partial x^2} = 0\,,$$

describing small amplitude transverse waves on a uniform string of mass per unit length ρ and tension T. Derive also a formula for the wave speed c. You may assume that $\partial y/\partial x$ is small and that gravity can be neglected. [4]

(b) Show that the energy transported per unit time by a sinusoidal wave on a uniform string is

$$P = \frac{1}{2} T \omega k A^2,$$

where ω is the angular frequency of the wave, k its wavenumber and A its amplitude. [5]

(c) A string of mass density per unit length ρ_1 is connected to a second string of mass density per unit length ρ_2, where $\rho_2 = 4\rho_1$. A sinusoidal wave propagates along the lighter string towards the boundary between the strings, where some of the wave is transmitted into the more massive string, and some is reflected.

Obtain expressions for the amplitudes of the transmitted and reflected waves, and hence find the values of the transmission and reflection coefficients. Explain your choice of boundary conditions. [4]

(d) Use the result from part (b) to find: (i) the ratio of the transmitted to incident power; and (ii) the ratio of the reflected to incident power, in the string system from part (c). Show that the sum of the reflected and transmitted power equals the incident power. [3]

(e) Describe what would happen if (i) $\rho_1 = \rho_2$, (ii) $\rho_1 \ll \rho_2$, or (iii) $\rho_1 \gg \rho_2$, in the string system of part (c). Give an interpretation of the sign of the reflection and transmission coefficients in these cases. [4]

12. (a) Show by explicit calculation that $\nabla \times \mathbf{F} = 0$ when

$$\mathbf{F} = 2yz^2\,\hat{\mathbf{i}} + 2xz^2\,\hat{\mathbf{j}} + 4xyz\,\hat{\mathbf{k}}\,.$$

[2]

(b) Given the result from part (a), discuss why we expect $\mathbf{F} \cdot \mathrm{d}\mathbf{r}$ to be an exact differential of a scalar function $\phi(x, y, z)$, where $\mathrm{d}\mathbf{r}$ is a vector line element, and obtain an expression for ϕ. [4]

(c) Evaluate $\int \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ by integrating along the following two paths:

i) along the curve $x = t^2$, $y = 2t$, $z = t^3$, where t varies from $t = 0$ to $t = 1$;

ii) along a path that proceeds first along the x-axis, then the y-axis, then the z-axis, from the origin to the point $(1, 2, 1)$.

[4]

(d) Compare the results from part (c) with the difference in the value of ϕ at the two ends of each path and comment on your results. [4]

(e) State Stokes' theorem for the case of a vector field $\mathbf{A}$ over a surface S with a closed boundary C. Verify Stokes' theorem for the case where

$$\mathbf{A} = (2x - y)\hat{\mathbf{i}} - yz^2\hat{\mathbf{j}} - y^2z\hat{\mathbf{k}}$$

where S is the upper half surface ($z > 0$) of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary in the xy plane. [6]