A10285W1

FIRST PUBLIC EXAMINATION

Trinity Term

Preliminary Examination in Physics

Paper CP3: MATHEMATICAL METHODS 1

also

Preliminary Examination in Physics and Philosophy

Wednesday 9th September 2015, 9.30 am – 12.00 midday

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

The use of calculators is **not** *permitted.*

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. Let z be a complex number. Sketch on separate Argand diagrams the set of all possible solutions for z for each of the following cases:

(i) $3z + z^* = 1 + \mathrm{i}$.

(ii) $z^4 = 4\mathrm{i}$.

(iii) $|z - 2| = \mathrm{Re}(z)$.

Where z^* is the complex conjugate of z and Re indicates the real part. [6]

2. State de Moivre's theorem and use it to prove that

$$\cos^3 x = a\cos(3x) + b\cos(x)\sin^2(x),$$

stating the required values of the constants a and b. [4]

3. Determine whether the following matrices are orthogonal, unitary, Hermitian, or none of these (note that some matrices may be more than one of these):

$$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1 & 0 & -\sqrt{3} \\ 0 & 1 & 0 \\ \sqrt{3} & 0 & 1 \end{pmatrix}.$$

[5]

4. Diagonalise the matrix

$$\begin{pmatrix} 5 & 0 & -2 \\ 0 & -3 & 0 \\ -2 & 0 & 2 \end{pmatrix},$$

and find its normalised eigenvectors. [7]

5. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be vectors in three-dimensional real space $\mathcal{R}^3$. Show that

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = -\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}).$$

[3]

6. Prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\log\left(\sec t + \tan t\right)\right) = \sec t.$$

Hence find the solution to the differential equation

$$\frac{dx}{dt} + 3x\tan t = \cos^2 t,$$

given that $x = 2$ at $t = 0$. [6]

7. State whether each of the following sets of vectors forms a basis in the corresponding stated vector space.

(i) $\begin{pmatrix}4\\0\end{pmatrix}$ and $\begin{pmatrix}1\\-1\end{pmatrix}$ in $\mathcal{R}^2$.

(ii) $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-1\end{pmatrix}$ in $\mathcal{R}^3$.

(iii) $\begin{pmatrix}1\\1\\2\end{pmatrix}$, $\begin{pmatrix}1\\-2\\0\end{pmatrix}$ and $\begin{pmatrix}2\\-1\\2\end{pmatrix}$ in $\mathcal{R}^3$

(iv) $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1+\mathrm{i}\end{pmatrix}$ in $\mathcal{C}^2$.

where $\mathcal{R}^n$ is n-dimensional real space and $\mathcal{C}^n$ is n-dimensional complex space.

Give reasons for your answers. [5]

8. Let $\mathbf{M}$ be a $n \times n$ real symmetric matrix. What are the conditions required of the eigenvalues of $\mathbf{M}$ in order that $\mathbf{M^2} = \mathbf{M}$? [4]

Section B

9. Find the general solutions to the following differential equations.

(a)

$$2\frac{\mathrm{d}y}{\mathrm{d}x} = (x+y)^2,$$

(b)

$$y\frac{\mathrm{d}y}{\mathrm{d}x} = x + \frac{2y^2}{x},$$

(c)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ay + \frac{e^{-2x}}{y^b}.$$

[20]

10. A mass attached to a spring executing driven, damped, simple harmonic motion can be described by the following differential equation:

$$\ddot{z} + \gamma\dot{z} + (k/m)z = (F_0/m)e^{\mathrm{i}\omega t}.$$

Describe the physical origin of each term. Under what conditions does the auxiliary equation (with $F_0 = 0$) show oscillating solutions? [3]

Show that the steady-state solution of this equation can be written as a complex number in the form $re^{\mathrm{i}(\omega t - \phi)}$ where

$$r = \frac{F_0/m}{\left[\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2\omega^2\right]^{\frac{1}{2}}},$$

$$\phi = \arctan\left\{\frac{\gamma\omega}{\left(\omega_0^2 - \omega^2\right)}\right\},$$

and $\omega_0^2 = k/m$. [6]

When this system is subjected to a constant periodic force $[F(t) = F_0\cos(\omega t)]$ what is the average power, W_{av}, transferred? Show that the maximum occurs at $\omega = \omega_0$ and that this has the value

$$W_{\mathrm{av}} = \frac{F_0^2}{2\,m\,\gamma}.$$

[7]

Sketch a graph of the power against angular frequency and mark the half-power points at $\omega_1$and ω_2. For a sharp resonance show that $|\omega_1 - \omega_2| = \gamma$. [4]

11. The matrices $\mathbf{A}$ and $\mathbf{B}$ are defined as

$$\mathbf{A} = \begin{pmatrix} 1 & 6 & -2 \\ -1 & 2 & 1 \\ 0 & 6 & -1 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 1 & 1 & 4/3 \\ 0 & 1/2 & 1/3 \\ 1 & 1 & 1 \end{pmatrix}$$

and the inverse of $\mathbf{B}$ is

$$\mathbf{B}^{-1} = \begin{pmatrix} -1 & -2 & 2 \\ -2 & 2 & 2 \\ 3 & 0 & -3 \end{pmatrix}.$$

The matrix $\mathbf{D}$ is defined by

$$\mathbf{D} = \mathbf{B}^{-1}\mathbf{A}\mathbf{B}.$$

(a) Show that $\mathbf{D}$ is diagonal of the form

$$\mathbf{D} = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}$$

and find the values of d_1, d_2, and d_3. [6]

(b) Write down an expression for $\mathbf{D}^n$, where n is a positive integer, in terms of d_1, d_2, and d_3, and find an expression for $\mathbf{A}^n$ in terms of $\mathbf{D}^n$, $\mathbf{B}$, and $\mathbf{B}^{-1}$. [6]

(c) Use the definition of matrix $\mathbf{D}$ to find the inverse of matrix $\mathbf{A}$. [4]

(d) Show that the determinant of $\mathbf{A}^n$ is $(d_1 d_2 d_3)^n$. Verify this relationship explicitly in the $n = 1$ case. [4]

12. Let $f(x)$ and $g(x)$ be real functions of a real parameter x. Let the scalar product of f and g be defined by

$$\langle f|g\rangle = \int_{-\pi}^{\pi} f(x)g(x)dx\,.$$

Consider the family of functions

$$f_n(x) = \sin(nx)$$

where n is a non-negative integer parameter.

(a) Find the value of $\langle f_n(x)|f_m(x)\rangle$ in the cases

(i) $m = n$

(ii) $m \neq n$. [8]

(b) The family of functions $f_n(x)$ forms a basis in the vector space of real odd functions defined inside the interval $(-\pi, \pi)$. Explain why the basis is orthogonal. [2]

By considering the scalar product

$$\langle h(x)|f_m(x)\rangle$$

express the function

$$h(x) = x$$

in that basis, by finding the real numbers a_n such that

$$h(x) = \sum_{n=1}^{\infty} a_n f_n(x)$$

inside the interval $(-\pi, \pi)$. [8]

(c) Why is it not possible to write the function $h_2(x) = x^2 - 1$ in this basis? [2]