A10285W1

FIRST PUBLIC EXAMINATION

Trinity Term

Preliminary Examination in Physics

Paper CP3: MATHEMATICAL METHODS 1

also

Preliminary Examination in Physics and Philosophy

Wednesday 10th June 2015, 2.30 pm – 5.00 pm

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

The use of calculators is **not** *permitted.*

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. Draw an Argand diagram, with labelled values, showing the locus of points that satisfy $|z| = \arg(z)$. [2]

2. Find the values of the complex number z that satisfy $z = \mathrm{i}^{\mathrm{i}}$. [3]

3. Find the solution to the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y\cos x = \sin(2x).$$

[4]

4. Given the matrix

$$\mathbf{M} = \frac{1}{2}\begin{pmatrix} \cosh y & \sinh y \\ \sinh y & \cosh y \end{pmatrix},$$

where y is a real parameter, find the elements of $\mathbf{M}^4$. [6]

5. Let $\mathbf{A}$ and $\mathbf{B}$ be Hermitian matrices. Which additional nontrivial conditions, if any, are necessary for the following statements to be true?

(i) $\mathbf{AB}$ is Hermitian

(ii) $\mathrm{i}(\mathbf{AB} - \mathbf{BA})$ is Hermitian

(iii) The trace of $\mathbf{A}$ is real

(iv) The eigenvectors of $\mathbf{A}$ are mutually orthogonal.

Give reasons for your answers. [6]

6. Let $\mathbf{x}$ be a vector in two-dimensional real space. By finding the eigenvalues and eigenvectors of $\mathbf{M}$, or otherwise, sketch the locus of points $\mathbf{x}$ that satisfy the equation

$$\mathbf{x}^{\mathrm{T}}\,\mathbf{M}\,\mathbf{x} = 4$$

given that

$$\mathbf{M} = \begin{pmatrix} 5 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}.$$

[6]

7. Find the volume of the tetrahedron which has vertices at $(0, 0, 1)$, $(1, 2, -1)$, $(0, 1, 4)$ and $(-1, -1, 1)$. [4]

8. Let $\mathbf{A}$ be an invertible $n \times n$ real matrix, and c be a real number. Express the following quantities in terms of $\det(\mathbf{A})$.

(i) $\det(\mathbf{A}^{\mathrm{T}})$

(ii) $\det(\mathbf{A}^{-1})$

(iii) $\det(c\mathbf{A})$

(iv) $\det(\mathbf{A}^{c})$ [4]

9. Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be vectors in three-dimensional real space. Find, in terms of $\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle$, the value of the scalar triple product $\langle \mathbf{A}, \mathbf{B}, \mathbf{C} \rangle$, where

$$\mathbf{A} = \frac{\mathbf{b} \times \mathbf{c}}{\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle}, \quad \mathbf{B} = \frac{\mathbf{c} \times \mathbf{a}}{\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle}, \quad \text{and} \quad \mathbf{C} = \frac{\mathbf{a} \times \mathbf{b}}{\langle \mathbf{a}, \mathbf{b}, \mathbf{c} \rangle}.$$

[5]

Section B

10. **(a)** Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be non-zero vectors in n-dimensional real space, $\mathbb{R}^n$. Let S be the set of points $\mathbf{x}$ in $\mathbb{R}^n$ given by the equation

$$\mathbf{x} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c},$$

for any values of the real scalars λ and μ. Describe the possible geometric shapes of S, and the conditions on $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ under which they occur, when:

(i) $n = 1$

(ii) $n = 2$

(iii) $n = 3$.

In the case where $n = 3$ and $\mathbf{b} = \mathbf{c}$, find the shortest distance from S to the point with position vector $\mathbf{p}$. [8]

(b) Consider the set Q of points $\mathbf{x}$ in $\mathbb{R}^3$ which simultaneously satisfy the two equations

$$|\mathbf{x}| = \alpha$$

and

$$|\mathbf{x} - \mathbf{d}| = \beta,$$

where $\mathbf{d}$ is a fixed position vector.

Describe fully the possible shapes and locations of Q, and the conditions on the real parameters α and β which distinguish the different cases. [9]

Discuss briefly how these results might generalise to the n-dimensional case. [3]

11. Consider the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\gamma\frac{\mathrm{d}x}{\mathrm{d}t} + \Omega^2 x = f(t),$$

where x and t are real variables, γ and Ω are positive real constants satisfying $\Omega > \gamma$,

$$f(t) = \begin{cases} A\sin\omega t & \text{for } t \leq 0 \\ 0 & \text{for } t > 0, \end{cases}$$

and A and ω are real constants.

Given that x is bounded for all t, find $x(t)$ for

(i) $t \leq 0$ [8]

(ii) $t > 0$. [8]

Use this example to briefly explain the meaning of terms *bandwidth* and *quality factor* in the context of a damped harmonic oscillator. [4]

12. Consider the three matrices

$$\boldsymbol{\sigma}_{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_{(2)} = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\sigma}_{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(i) Find a general expression for

$$\boldsymbol{\sigma}_{(j)}\boldsymbol{\sigma}_{(k)} - \boldsymbol{\sigma}_{(k)}\boldsymbol{\sigma}_{(j)}.$$

[4]

For any complex two-vector $\mathbf{v}$ let the components x_j of an associated three-vector $\mathbf{x}$ be obtained from $\mathbf{v}$ by the association

$$x_j = \tfrac{1}{2}\mathbf{v}^\dagger \boldsymbol{\sigma}_{(j)} \mathbf{v}.$$

(ii) Find the value of $\mathbf{x}$ for $\mathbf{v} = (a, b)$, where a and b are complex numbers. [3]

Let $\mathbf{v}' = \mathbf{U}\mathbf{v}$, where

$$\begin{aligned} \mathbf{U} &= \exp\left(\mathrm{i}\mathbf{T}/2\right) \\ &= 1 + \mathrm{i}\mathbf{T}/2 + \tfrac{1}{2}\left(\mathrm{i}\mathbf{T}/2\right)^2 + \ldots, \end{aligned}$$

where $\mathbf{T}$ is defined by

$$\mathbf{T} = \alpha_1\boldsymbol{\sigma}_{(1)} + \alpha_2\boldsymbol{\sigma}_{(2)} + \alpha_3\boldsymbol{\sigma}_{(3)},$$

and where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ is a real three-vector.

(iii) Given the general matrix identity that when $\mathbf{AB} = \mathbf{BA}$ it follows that

$$\exp\left(\mathbf{A}\right)\exp\left(\mathbf{B}\right) = \exp\left(\mathbf{A} + \mathbf{B}\right),$$

show that $\mathbf{U}$ is a unitary matrix. [2]

(iv) By considering the series expansion for $\mathbf{U}$, show that eigenvectors of $\mathbf{T}$ are also eigenvectors of $\mathbf{U}$. [1]

(v) By considering the eigenvalues of $\mathbf{T}$, or otherwise, find the eigenvalues of $\mathbf{U}$. [4]

(vi) Find the eigenvectors of $\mathbf{U}$ when $\alpha_1 = \alpha_2 = 0$. [2]

(vii) Hence find the nature of the transformation $\mathbf{x} \to \mathbf{x}'$ which corresponds to $\mathbf{v} \to \mathbf{v}' = \mathbf{U}\mathbf{v}$ for $\alpha_1 = \alpha_2 = 0$. [4]

13. (a) A real n-dimensional vector $\mathbf{x}_{(i+1)}$ is defined iteratively in terms of its previous value $\mathbf{x}_{(i)}$ by the expression

$$\mathbf{x}_{(i+1)} = \mathbf{M}\,\mathbf{x}_{(i)} + \mathbf{c},$$

where $\mathbf{M}$ is a real $n \times n$ matrix, and $\mathbf{c}$ is a real n-dimensional vector.

(i) By considering $\delta\mathbf{x}_{(k)} = \mathbf{x}_{(k+1)} - \mathbf{x}_{(k)}$, or otherwise, find the conditions on $\mathbf{M}$ and $\mathbf{c}$ that guarantee the existence of a constant solution

$$\mathbf{x}_{(i)} = \mathbf{X} \quad \text{for all } i,$$

and find that $\mathbf{X}$ in terms of $\mathbf{M}$ and $\mathbf{c}$.

(ii) The k^{th} value of $\mathbf{x}$ can be written

$$\mathbf{x}_{(k)} = f(\mathbf{M})\,\mathbf{x}_{(1)} + g(\mathbf{M})\,\mathbf{c}$$

for $k \geq 2$. Find the functions $f(\mathbf{M})$ and $g(\mathbf{M})$.

(iii) Given that $\mathbf{M}$ is diagonalisable, what are the conditions required of its eigenvalues $\lambda_1, \ldots, \lambda_n$ in order that

$$\lim_{i\to\infty} \left(\mathbf{x}_{(i)}\right) = \mathbf{X}$$

for an arbitrary $\mathbf{x}_1$? [10]

(b) Let $\mathbf{v}$ be a n-dimensional real vector and $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ real symmetric matrices. The quantities a and b are defined by

$$a = \tfrac{1}{2}\left(\mathbf{v}^{\mathrm{T}}\mathbf{A}\,\mathbf{v} - \alpha|\mathbf{v}|^2\right),$$
$$b = \tfrac{1}{2}\left(\mathbf{v}^{\mathrm{T}}\mathbf{B}\,\mathbf{v} - \beta|\mathbf{v}|^2\right) + a.$$

(i) By considering the derivatives $\frac{\partial a}{\partial v_i}$ for each component v_i of $\mathbf{v}$, find the values of a which are stationary with respect to small changes in all n components of $\mathbf{v}$. Find the value(s) of $\mathbf{v}$ at which those stationary values of a are found.

(ii) Under what conditions will a and b simultaneously be stationary at the same value of $\mathbf{v}$? [10]