4296

**FIRST PUBLIC EXAMINATION**

**Trinity Term**

**Preliminary Examination in Physics**

**Paper CP4: MATHEMATICAL METHODS 2**

**also**

**Preliminary Examination in Physics and Philosophy**

**Thursday 11 September 2014, 9.30 am – 12 pm**

**Time allowed: $2\frac{1}{2}$ hours**

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*Answer* **all** *of Section A and* **three** *questions from Section B.*

*The use of calculators is permitted.*

*A list of physical constants, mathematical formulae and conversion factors accompanies this paper.*

*The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.*

**Do NOT turn over until told that you may do so.**

## Section A

**1.** State the divergence (Gauss') theorem. Evaluate the surface integral

$$\int_S \mathbf{r} \cdot \mathbf{dS}\,,$$

where $\mathbf{r}$ is a position vector and $S$ is a closed surface that encloses a volume $V$. [4]

**2.** Expand the following functions of two variables as a series about $x, y = 0$, giving at least the four lowest-order, non-zero terms:

(a) $\cos x \sinh y$ , [2]

(b) $\dfrac{\ln(1+x)}{1+y}$ . [2]

**3.** State Stokes' theorem.

A vector $\mathbf{A}$ is defined as $\mathbf{A} = (x^2 - y^2)\hat{\mathbf{i}} + 2xy\hat{\mathbf{j}}$.

(a) Find $\nabla \times \mathbf{A}$.

(b) Evaluate the integral $I$, where

$$I = \int (\nabla \times \mathbf{A}) \cdot \mathrm{d}\mathbf{S}\,,$$

over a rectangle defined on a surface $\mathbf{S}$ in the $(x, y)$ plane that is bounded by the lines $x = 0$, $x = a$, $y = 0$, $y = b$.

(c) Evaluate the line integral $\int \mathbf{A} \cdot \mathrm{d}\mathbf{r}$ around the boundary of the same rectangle, and thus verify Stokes' theorem in this case. [6]

**4.** (a) Using series expansions, evaluate the limit

$$\lim_{x \to 0} \frac{1 + x^2 - \exp\left(x^2\right)}{x \sin x - x^2}\,.$$

[2]

(b) A function $I(x)$ is defined by the integral

$$I(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left(-t^2\right) dt\,.$$

Compute the first two non-zero terms of the Taylor series of $I(x)$ around $x = 0$. [4]

**5.** Using the change of variables $u = x - 3y$ and $v = 2x + y$, evaluate

$$\iint_D \frac{x - 3y}{2x + y}\, \mathrm{d}x\, \mathrm{d}y\,,$$

where $D$ is the parallelogram bounded by the lines $y = -2x + 1$, $y = -2x + 4$, $y = x/3$, and $y = (x - 7)/3$. [6]

**6.** State the general criteria for determining the nature of the stationary values of functions of two variables. [4]

Find the stationary point of the function

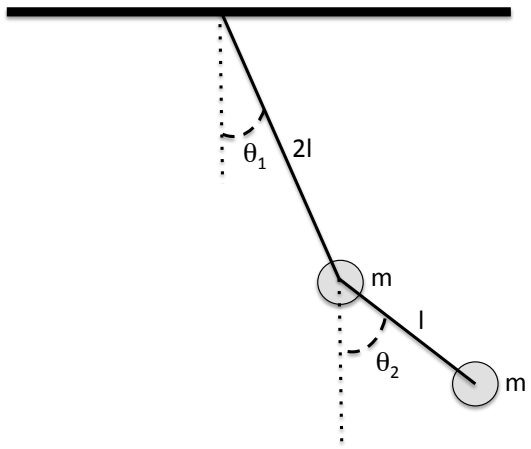
$$f(x, y) = x^2 - y^2 + 2x - 4y + 10$$

and determine its nature. [4]

**7.** Two sinusoidal waves of equal amplitude, with angular frequencies $\omega + \epsilon$ and $\omega - \epsilon$, propagate along the positive $x$-direction in a dispersive medium, with corresponding wavenumbers $k + \delta$ and $k - \delta$. Write down an expression for the sum of the two waves and use it to define the group velocity of the wavepacket, explaining its physical significance. Show that the group velocity would be equal to the phase velocity, if the medium were non-dispersive. [6]

# Section B

**8.** Consider a double pendulum, comprising two identical masses $m$, suspended using two massless rods. The first rod, of length $2l$, is attached to a fixed pivot point in the ceiling, while the second rod, of length $l$, is attached to the first mass, allowing frictionless rotation of the system in the vertical plane, as shown in the figure below.




(a) Taking the zero of potential energy to be at the equilibrium position, derive an expression for the potential energy of the system, $U$, in terms of the respective angles $\theta_1$ and $\theta_2$ between the two rods and the vertical direction, where the angles may be assumed to be small. [2]

(b) Derive an expression for the kinetic energy of the system, $K$, in terms of $\theta_1$ and $\theta_2$, where the angles may again be assumed to be small. [2]

(c) Use the vector notation

$$\boldsymbol{\theta} = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

to rewrite the expressions as $U = \frac{1}{2}\boldsymbol{\theta}^{\mathbf{T}}\mathbf{V}\boldsymbol{\theta}$ and $K = \frac{1}{2}\dot{\boldsymbol{\theta}}^{\mathbf{T}}\mathbf{T}\dot{\boldsymbol{\theta}}$, where $\boldsymbol{\theta}^{\mathbf{T}}$ denotes the transpose of $\boldsymbol{\theta}$ and where $\dot{\boldsymbol{\theta}}$ denotes the time derivative of $\boldsymbol{\theta}$. Derive expressions for the matrices $\mathbf{V}$ and $\mathbf{T}$. [6]

(d) Write down an expression for the total energy, $E = U + K$, and use the conservation of total energy, $\frac{\mathrm{d}E}{\mathrm{d}t} = 0$, to derive the equations of motion of the system in matrix form. [4]

(e) Find the frequencies of the normal modes of oscillation of the system. [6]

**9.** (a) A sphere of radius $R$ with centre at the origin is cut by two parallel planes at $z = \pm a$, where $a < R$. Write, in cylindrical coordinates, a triple integral which gives the volume of that part of the sphere between the two planes, and evaluate that integral. [7]

(b) A multiple integral $I$ is defined by

$$I = \int_0^\infty \int_0^\infty e^{-(x+y)} \sqrt{xy}\, dxdy\,.$$

The coordinates $(x, y)$ may be expressed in terms of a new set of variables $(u, v)$, defined by

$$x = \frac{u}{2}(1+v), \quad y = \frac{u}{2}(1-v)\,.$$

Find the Jacobian of the transformation from $(x, y)$ to $(u, v)$ coordinates.

By considering the combinations $x \pm y$ and $xy$, find the region of the $(u, v)$ plane onto which the positive quadrant $x \geq 0$, $y \geq 0$ of the $(x, y)$ plane maps and draw a diagram illustrating that $(u, v)$ region. Hence evaluate the integral $I$. [13]

$[\int_0^\infty u^n \exp(-u)du = n!]$

**10.** An elastic, horizontal string, of tension $T$ and mass per unit length $\rho$, is held fixed at both ends, where $x = 0$ and $x = L$. At time $t = 0$, the string is displaced transversely along the $y$ direction in such a way that:

$$y(x, 0) = \sin\frac{\pi x}{L} - \frac{1}{3}\sin\frac{3\pi x}{L}\,.$$

(a) Calculate the total energy of the string at $t = 0$. [5]

(b) The string is now released and starts to oscillate. Derive the wave equation

$$\frac{\partial^2 y(x,t)}{\partial t^2} - c^2\frac{\partial^2 y(x,t)}{\partial x^2} = 0$$

describing small amplitude transverse waves on the string, and a formula for the wave speed $c$. [5]

(c) Solve this wave equation to obtain the transverse displacement $y(x, t)$ of the string at time $t$. [4]

(d) Under what conditions would the string return, at some later time, to its original shape? At what time would this happen? [2]

(e) Calculate the kinetic energy associated with each normal mode, found to be a solution of the wave equation in part (c), as a function of time $t$. [4]

**11.** Consider a vector field $\mathbf{v}$ such that $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, where $\boldsymbol{\omega}$ is a constant vector and $\mathbf{r}$ is the usual vector indicating spatial position.

(a) Show that $\boldsymbol{\omega} = \frac{1}{2}\nabla \times \mathbf{v}$. [5]

(b) Suppose that $\mathbf{v}$ describes the velocity field of a moving fluid. Describe the subsequent motion and rotation of an object of finite size, after being placed in this fluid, neglecting any motion due to the force of gravity. [2]

(c) A field $\mathbf{F}$ is called irrotational if curl $\mathbf{F} = 0$. Find constants $a$, $b$ and $c$ such that

$$\mathbf{F} = (x + 3y + az)\hat{\mathbf{i}} + (bx + y - z)\hat{\mathbf{j}} + (3x + cy - 2z)\hat{\mathbf{k}}\,,$$

is irrotational. [4]

(d) Show that in this case, $\mathbf{F}$ can be expressed as the gradient of a scalar function $\phi$ and derive the exact form of $\phi$ as a function of $x$, $y$ and $z$. [5]

(e) Show that this particular $\phi$ is also a solution of Laplace's equation ($\nabla^2\phi = 0$). What is the consequence for the divergence of the field $\mathbf{F}$? [4]